Maintenance policies based on time-dependent repair cost limits

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This paper considers two replacement policies for systems which, during their useful life, are subject to deterioration. Strategy 1: after a failure, the repair cost is estimated. If the repair cost exceeds a given limit, the system is not repaired, but replaced with a new one. So far, this policy has been analysed only for constant repair cost limits. This paper investigates the effect of applying time-dependent repair cost limits on the long-run maintenance cost rate. Examples show that, compared to the application of constant repair cost limits, a reduction of the maintenance cost rate between 5% and 10% can be expected. Strategy 2: the system is replaced as soon as the total repair costs arising during its running time exceed a given limit. Compared to the economic lifetime method, which is based on the average repair cost development and that requires the same data input, maintenance cost savings up to 20% could be achieved.

Introduction

Technical systems are subject to deterioration with usage and age. For safety and cost reasons, breakdowns or malfunctions have to be prevented as far as possible by application of maintenance strategies, which are tailored to the respective situation. A particularly important decision is when to replace a system by a new one. Numerous replacement strategies have been suggested and analysed so far. They aim at minimizing loss costs and/or maximizing the system availability. Replacement policies based on limits on individual repair costs or total repair costs have proved to be particularly user-friendly and efficient. Based on individual repair costs, the decision whether to replace or not is made as follows:

Strategy 1. After a system failure, the necessary repair cost is estimated. The system is replaced by an equivalent new one if the repair cost exceeds a given level $c(t)$, where t is the age of the system at the time of failure. Otherwise, a minimal repair is carried out. By definition, a minimal repair does not affect the failure rate of the system, i.e. the failure rate of the system after a repair has the same value as at the time of failure.

Drinkwater and Hastings[1] were the first to propose the application of repair cost limits. They and later Koshimae *et al.*,[2] Love and Guo,[3] and Mahon and Bailey[4] analysed repair cost limit replacement strategies for piecewise constant repair cost limit functions $c(t)$. Whereas these authors used heuristically based methods, Beichelt and Fischer[5] were the first to present a mathematically exact approach under the same assumptions. The case of a constant repair cost limit, i.e. $c(t) \equiv c$, has been considered by, among others, Kapur *et al.*[6] and Park.[7] For recent summaries, see Nkadimeng[8] and Beichelt.[9]

System ageing implies a progressive increase in the mean failure frequency and in the mean repair costs with increasing system age t. Thus, applying a continuously decreasing repair cost limit function $c(t)$ within strategy 1 can be expected to yield lower average maintenance costs than applying a constant repair cost limit or a piece-wise constant repair cost limit function. In what follows, this conjecture is verified for two special functions $c(t)$.

A disadvantage of strategy 1 is that the replacement decision is based solely on the cost of a single repair. Long-lasting situations characterized by a high failure intensity with the corresponding repair costs being under the limit do not induce a replacement, although the total repair cost during such a time period may justify a replacement. Hence, pursuing the following replacement strategy might be in many cases more appropriate.

Strategy 2. The system is replaced as soon as the total repair cost spent on it, since its installation, exceeds a given limit c.

Other advantages to strategy 1 are: strategy 2 is purely cost-based. No lifetime data need be known. Apart from repair costs, strategy 2 can take into account costs due to continuous monitoring, servicing, stock keeping, personnel costs, interest rates and so on.

Basic notation

X	System lifetime, a random variable
$F(t), \bar{F}(t)$	Distribution function, survival function of X: $\bar{F}(t) = 1 - F(t), t \geq 0$
$f(t)$	Probability density X
$\lambda(t)$	Failure rate of X: $\lambda(t) = f(t)/\bar{F}(t)$
$\Lambda(t)$	Integrated failure rate: $\int_0^t \lambda(x)dx$
X_t	Residual lifetime of a system after time t given that it has survived interval $[0, t]$
$F_t(x)$	Distribution function of X_t
$\bar{F}_t(x)$	Survival function of X_t: $\bar{F}_t(x) = 1 - F_t(x)$
c_m, c_r	Mean cost of a minimal repair, a replacement, $c_m < c_r$

Basic assumptions

1) Maintenance actions comprise minimal repairs and replacements (strategy 1).
2) After a replacement, a system is 'as good as new'.
3) All maintenance actions take only negligible time.
4) The system is ageing. (Its failure rate is increasing.)

A minimal repair, carried out after a system failure, restores the capability of the system to continue its work, but does not affect its failure rate (hazard rate) of the system. Thus, if a minimal repair is done at system age t ('age' refers to the time point of opening the system), then its residual lifetime has distribution function and survival function

$$F_t(x) = \frac{F(t+x) - F(t)}{\bar{F}(t)}; \quad \bar{F}_t(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}; \quad t, x \geq 0. \qquad (1)$$

It is well known that the time points $\{X_1, X_2, ...\}$ of successive minimal repairs (if not interrupted by a replacement) are governed by a nonhomogeneous Poisson process with intensity function $\lambda(t)$.

The most common criterion for evaluating maintenance policies is the maintenance cost per unit time K on condition that the maintenance process continues to infinity (i.e. over a sufficiently long time period). If L denotes the random length of

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a replacement cycle (time between two neighbouring replacements) and C the random maintenance cost per cycle, then

$$K = \frac{E(C)}{E(L)}.$$

In this paper, K is referred to as the *maintenance cost rate.*

Replacement strategy 1

To be able to analyse this strategy mathematically, some basic results concerning a two-failure mode model need to be summarized (for details see Beichelt[1]). A system has the property that its failures can be classified in the following way:

1. *Mode 1 failures*: failures of this type are (and can be) removed by minimal repairs (minor failures).
2. *Mode 2 failures*: failures of this type are removed by replacements (major failures).

A failure, which occurs at system age t, is a mode 2 failure with probability $p(t)$, and a mode 1 failure with probability $\bar{p}(t) = 1 - p(t)$. It is assumed that mode 1 and mode 2 failures occur independently of each other. Within this model, the running time of the system is partitioned into cycles, which are the times between neighboring replacements. The lengths of these cycles are independent, identically as Y distributed random variables with

$$G(t) = P(Y \le t) = 1 - \exp\{-\textstyle\int_0^t p(x)\lambda(x)dx\},$$
$$\bar{G}(t) = P(Y > t) = 1 - G(t). \tag{2}$$

The random number M of minimal repairs in a cycle has mean value

$$E(M) = \int_0^\infty \Lambda(t)dG(t) - 1. \tag{3}$$

Hence, the maintenance cost rate under the strategy 'maintenance according to the failure type' is

$$K = \frac{\left(\int_0^\infty \Lambda(t)dG(t) - 1\right)c_m + c_r}{\int_0^\infty \bar{G}(t)\,dt}. \tag{4}$$

In particular, on condition $p(t) \equiv p$,

$$\bar{G}(t) = 1 - G(t) = \left[\bar{F}(t)\right]^p, \qquad E(M) = (1-p)/p.$$

Therefore, the corresponding maintenance cost rate is

$$K = \frac{\left(\frac{1-p}{p}\right)c_m + c_r}{\int_0^\infty \left[\bar{F}(t)\right]^p dt}. \tag{5}$$

The two failure mode model is applicable to analysing strategy 1. The probabilities $p(t)$ and $\bar{p}(t)$ are generated by the probability distribution function of the repair costs $R_t(x)$. Let $C(t)$ denote the random repair caused by a failure at t time. Then, $R_t(x) = P(C(t) \le x)$ and a replacement will be carried out with probability

$$p(t) = \bar{R}_t(c(t)) = 1 - R_t(c(t)) = P(C(t) > c(t))$$

and a minimal repair with probability

$$\bar{p}(t) = R_t(c(t)) = P(C(t) \le c(t)).$$

Hence, by (2), the random length L of a replacement cycle has distribution function

$$G(t) = 1 - e^{-\int_0^t \bar{R}_x(c(x))\lambda(x)dx}$$

and mean value

$$E(L) = \int_0^\infty e^{-\int_0^t \bar{R}_x(c(x))\lambda(x)dx}dt. \tag{6}$$

From (3), the mean number of minimal repairs in a cycle is

$$E(M) = \int_0^\infty \Lambda(t)\lambda(t)\bar{R}_t(c(t))e^{-\int_0^t \bar{R}_x(c(x))\lambda(x)dx}dt - 1. \tag{7}$$

Thus, by (4), the maintenance cost rate belonging to strategy 1 is

$$K_1 = \frac{\left[\int_0^\infty \Lambda(t)\lambda(t)\bar{R}_t(c(t))e^{-\int_0^t \bar{R}_x(c(x))\lambda(x)dx}dt - 1\right]c_m + c_r}{\int_0^\infty \bar{R}_t(c(t))e^{-\int_0^t \bar{R}_x(c(x))\lambda(x)dx}dt}. \tag{8}$$

Example 1 (*constant repair cost limit*). For the sake of comparison, a constant repair cost limit together with a time-independent repair cost distribution is next considered:

$$c(t) \equiv c, \qquad R_t(x) \equiv R(x).$$

Then, from (5),

$$E(L) = \int_0^\infty \left[\bar{F}(t)\right]^{\bar{R}(c)} dt,$$

$$E(M) = R(c)/\bar{R}(c).$$

In addition, it will be assumed throughout the paper that the system lifetime has a Weibull distribution with distribution function

$$F(t) = 1 - e^{-\lambda t^\delta}, \quad t \ge 0, \ \delta > 1, \ \lambda > 0. \tag{9}$$

Then the mean cycle length is

$$E(L) = \Gamma(1 + 1/\delta)\left[\lambda\bar{R}(c)\right]^{-1/\delta}.$$

Now, formula (5) yields the corresponding maintenance cost rate:

$$K_1(c) = \frac{\frac{R(c)}{\bar{R}(c)}c_m + c_r}{\Gamma(1+1/\delta)\left[\lambda\bar{R}(c)\right]^{-1/\delta}}. \tag{10}$$

$K_1(c)$ depends on c only via $R(c)$. The value of $y = \bar{R}(c)$ minimizing $K_1(c)$ is easily seen to be

$$y* = \bar{R}(c*) = \frac{\delta - 1}{k - 1},$$

where $k = c_r/c_m$. Note that, by assumption, $k > 1$ and $\delta > 1$. Hence, since $0 > y^* > 1$, an additional assumption has to be made:

$$1 < \delta < k. \tag{11}$$

Otherwise, the cost-optimal behaviour would be 'replace the system after every failure', i.e. $c^* = 0$. On condition (11), for any $\bar{R}(x)$ with inverse function $\bar{R}^{-1}(\cdot)$, the optimal repair cost limit $c = c^*$ is

$$c* = \bar{R}^{-1}\left(\frac{\delta - 1}{k - 1}\right). \tag{12}$$

Its application yields the smallest possible maintenance cost rate, which can be achieved with a constant repair cost limit:

$$K_1(c*) = \frac{\delta\lambda^{1/\delta}c_m}{\Gamma(1+1/\delta)}\left(\frac{k-1}{\delta-1}\right)^{1-1/\delta}.$$

If $\delta = 2$ (*Rayleigh distribution*), then $(1+1/\delta) = \Gamma(1.5) = \sqrt{\pi/4}$ so that

$$K_1(c^*) = 4c_m\sqrt{\frac{\lambda}{\pi}(k-1)} \approx 2.2568\, c_m\sqrt{\lambda(k-1)}. \tag{13}$$

Example 2 (Nkadimeng[8]). Let the repair cost limit function $c(t)$ be given by the decreasing function:

$$c(t) = \begin{cases} c_r, & 0 \le t < \dfrac{d}{c_r - c} \\ c + \dfrac{c}{d}, & \dfrac{d}{c_r - c} \le t < \infty \end{cases}, \quad 0 \le c < c_r.$$

In what follows, let the random cost of a repair C be independent of t and have a uniform distribution over the interval $[0, c_r]$, i.e.

$$R(x) = P(C \le x) = \begin{cases} x/c_r, & 0 \le x < c_r \\ 1, & c_r \le t < \infty \end{cases}.$$

Under this distribution, repair costs are restricted to the interval $[0, c_r]$. This property of a repair cost distribution makes sense, since a system failure causing repair costs which exceed replacement costs, will usually lead to a replacement.

Then a system failure at age t implies a replacement with probability

$$\overline{R}(c(t)) = \begin{cases} 0, & 0 \le t < \dfrac{d}{c_r - c} \\ \dfrac{c_r - c}{c_r} - \dfrac{d}{c_r t}, & \dfrac{d}{c_r - c} \le t < \infty \end{cases}, \quad 0 \le c < c_r.$$

Letting

$$r = d/c_r, \quad s = (c_r - c)/c_r, \quad \text{and} \quad z = r/s$$

yields

$$\overline{R}(c(t)) = \begin{cases} 0, & 0 \le t < z \\ s(1 - z/t), & z \le t < \infty \end{cases}. \tag{14}$$

Scheduling replacements based on (14) is well motivated: Replacements of systems in the first period of their useful life will not be scheduled. After this period, a failure makes a replacement more and more likely with increasing system age.

In what follows, let the system lifetime have the failure distribution (9) with $\delta = 2$ (Rayleigh distribution). Then,

$$\Lambda(t) = \lambda t^2, \quad \lambda(t) = 2\lambda t.$$

According to (7), the mean number of minimal repairs $E(M)$ within a cycle is

$$E(M) = 2\lambda^2 s \int_z^\infty t^3 \frac{t-z}{t} e^{-2\lambda s \int_z^t x \frac{x-z}{x} dx} dt - 1$$

$$= 2\lambda^2 s \int_0^\infty \left(x^3 + 2zx^2 + z^2 x\right) e^{-\lambda s x^2} dx - 1. \tag{15}$$

The three basic integrals in (15) have values

$$\int_0^\infty x^3 e^{-\lambda s x^2} dx = \frac{1}{2(\lambda s)^2},$$

$$\int_0^\infty x^2 e^{-\lambda s x^2} dx = \frac{1}{4\lambda s}\sqrt{\frac{\pi}{\lambda s}},$$

$$\int_0^\infty x e^{-\lambda s x^2} dx = \frac{1}{2\lambda s}.$$

Combining these integrals according to (15) yields a simple formula for $E(M)$:

$$E(M) = \frac{1-s}{s} + \sqrt{\frac{\lambda\pi}{s}}\, z + \lambda^2 z^2.$$

The mean lifetime of the system is

$$E(L) = z + \int_0^\infty e^{-\lambda s x^2} dx = z + \frac{1}{2}\sqrt{\frac{\pi}{\lambda s}}.$$

Since $z = r/s$, the corresponding maintenance cost rate is

$$K_1(r,s) = \frac{1}{r + \sqrt{\pi s/(4\lambda)}}\left(1 - s + \sqrt{\frac{\lambda\pi}{s}}\, r + \frac{\lambda}{s} r^2 + k\right) c_m.$$

In order to minimize $K_1(r,s)$ with respect to r and s, let us, in a first step, minimize $K_1(r,s)$ for fixed s with respect to r. The corresponding optimal value of r, denoted as $r^* = r^*(s)$, is solution of the quadratic equation $\partial K_1(r,s)/\partial r = 0$:

$$\left(r + \frac{1}{2}\sqrt{\frac{\pi s}{\lambda}}\right)^2 = \frac{s}{4\lambda}\left[4s(k-1) + 4 - \pi\right].$$

The right-hand side of this equation is positive, since, by assumption, $k = c_r/c_m > 1$. Thus, $r^* = r^*(s)$ is

$$r^*(s) = \sqrt{\frac{s}{4\lambda}}\left[\sqrt{4s(k-1) + 4 - \pi} - \sqrt{\pi}\right].$$

To make sure that $r^*(s) > 0$, an additional assumption has to be made:

$$k > \frac{\pi - 2}{2s} + 1. \tag{16}$$

The corresponding maintenance cost rate is

$$K_1(r^*(s)) = c_m\sqrt{\lambda}\sqrt{4(k-1) + \frac{4-\pi}{s}}. \tag{17}$$

Since $s \le 1$, $K_1(r^*(s))$ assumes its minimum at $s = 1$, so that $c = 0$. With $s = 1$ condition (16) holds if and only if $k > \pi/2 \approx 1.57$. Since replacement costs are usually much higher than repair costs (as a rule of thumb, $k = c_r/c_m > 20$), this condition hardly affects the practical application of the repair cost limit function (20).

Summarizing: If $k > \pi/2$, the optimal repair cost limit function and the corresponding maintenance cost rate are

$$c^*(t) = \begin{cases} c_r, & 0 \le t < \dfrac{d}{c_r - c} \\ \dfrac{d^*}{t}, & \dfrac{d^*}{c_r} \le t < \infty \end{cases}, \quad d^* = \frac{1}{2\sqrt{\lambda}}\left[\sqrt{4k - \pi} - \sqrt{\pi}\right] c_r, \tag{18}$$

$$K_1(d^*) = c_m\sqrt{\lambda(4k - \pi)}. \tag{19}$$

Comparison. Now let us compare $K_1(d^*)$ with the minimal maintenance cost rate (13), obtained under the same conditions as (19), but when applying the optimal constant repair cost limit c^* given by (12). One easily verifies that

$$K_1(c^*) = 4c_m\sqrt{\frac{\lambda}{\pi}(k-1)} < K_1(d^*) = c_m\sqrt{\lambda(4k - \pi)}$$

if and only if

$$k > \frac{16 - \pi^2}{16 - 4\pi} \approx 1.785.$$

This condition is slightly stronger than $k > \pi/2$; but, as pointed out previously, it hardly implies a restriction to practical applications. Hence, under the assumptions made, applying the optimal age-dependent decreasing repair cost limit function (18) is more efficient than applying the optimal constant repair cost limit (12).

Replacement strategy 2

Let $A(t)$ be the random total maintenance cost spent on a system, which was installed at $t = 0$, up to time point t, $t > 0$. Then $\{A(t), t \geq 0\}$ is a stochastic process. Its one-dimensional probability distribution is given by the family of distribution functions

$$\{F_{A(t)}(x) = P(A(t) \leq x),\ t \geq 0\}.$$

To simplify the discussion of strategy 2, the sample paths of $\{A(t), t \geq 0\}$ are assumed to be continuous and strictly increasing functions in t. Let $L(a)$ be the first passage time of $\{A(t), t \geq 0\}$ with regard to a positive level a, i.e.

$$L(a) = \min(t,\ A(t) = a).$$

Since $\{A(t), t \geq 0\}$ has increasing sample paths, the probability distribution of $L(a)$ is given by the probability distribution of $A(t)$:

$$F_{L(x)}(t) = P(L(x) \leq t) = P(A(t) > x) = 1 - F_{A(t)}(x) = \overline{F}_{A(t)}(x). \quad (20)$$

When replacing the system with a new one as soon as $A(t) = a$ (which is strategy 2), the corresponding maintenance cost rate is, by the strong law of large numbers,

$$K(a) = \frac{a + c_r}{E(L(a))}. \quad (21)$$

Example 3 Let the total repair cost $A(t)$ have distribution function

$$F_{A(t)}(x) = e^{-(\lambda t^2/x)^{\beta}};\ \lambda > 0,\ \beta > 0,\ x > 0\ \ (\textit{Frechet distribution}). \quad (22)$$

Then, from (20),

$$E(L(a)) = \int_0^{\infty} F_{A(t)}(a)\,dt = k_1\sqrt{a}, \quad (23)$$

where

$$k_1 = \sqrt{\frac{1}{\lambda}}\,\Gamma\left(\frac{1}{2\beta}+1\right).$$

Inserting (23) into (21) and minimizing with regard to a gives the optimal total repair cost limit a^* and the corresponding maintenance cost rate:

$$a^* = c_r, \quad K(a^*) = \frac{2}{k_1}\sqrt{c_r}.$$

It is interesting and quite intuitive that the optimal total repair cost limit coincides with the replacement cost. (However, other distribution parameters will not yield this result.)

If the system is always replaced after τ time units, then the corresponding maintenance cost rate is

$$K(\tau) = \frac{c_r + E(A(\tau))}{\tau}.$$

From (22),

$$E(A(\tau)) = \int_0^{\infty} \overline{F}_{A(\tau)}(x)\,dx = k_2\,\tau^2$$

with

$$k_2 = \lambda\,\Gamma\left(1 - \frac{1}{\beta}\right).$$

Then the optimal values of τ and $K(\tau)$ are

$$\tau^* = \sqrt{\frac{c_r}{2k_2}}, \quad K(\tau^*) = 2\sqrt{k_2\,c_r}.$$

Comparison. The inequality $K(a^*) > K(\tau^*)$ is equivalent to

$$1 < \Gamma\left(1+\frac{1}{\beta}\right)\left[\Gamma\left(1-\frac{1}{2\beta}\right)\right]^2.$$

It is easily seen that this inequality holds for all $\beta \geq 1$. Hence, applying replacement strategy 2 with the total repair cost limit $a = a^*$ is more efficient than applying the economic lifetime τ^*. Theoretically, if $\beta = 2$, then average cost savings between 33% and 6.2% can be expected. For other examples along this line, see Beichelt.[10]

Conclusions

Repair cost limit replacement policies (replacement strategy 1) are feasible maintenance policies for complicated systems. This paper investigates the effect of decreasing repair cost limits on the long-run maintenance cost rate. We show that for ageing systems the application of decreasing repair cost limits instead of constant repair cost limits may lead to a considerable reduction of the maintenance cost rate. Since the research done in this paper is restricted to Weibull distributed lifetimes and power distributed repair costs, more theoretical work needs to be done to explore the potentialities of an exact analytic treatment of age-dependent repair cost limits. Moreover, to be able to incorporate reliability requirements, further research should combine the repair cost limit approach with age or block replacement policies. Since in this paper repairs are assumed to be minimal, an important generalization would be to analyse repair cost limit policies in conjunction with general repairs.

In view of their simple structure and for not requiring information on system lifetime distributions, total repair cost limit replacement policies (replacement strategy 2) are particularly user-friendly. The fact that maintenance cost data are usually available facilitates their application for scheduling preventive replacements of complex, wear-subjected technical systems and for determining cost-optimal overhaul time points of whole industrial plants. The example analysed and many other ones provide strong arguments in favour of scheduling replacements on the basis of total repair cost limits instead of the economic lifetime. However, much more theoretical and practical work (simulation) needs to be done to get a deeper insight into the relationship between the total repair cost limit approach and the economic lifetime. As with replacement strategy 1, strategy 2 is not a suitable model if the reliability aspect of the underlying system is dominant.

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