Soft-Decision Decoding of Permutation Block Codes in AWGN and Rayleigh Fading Channels

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***Abstract*— We introduce an efficient soft-decision decoder for permutation block codes when assuming AWGN and slow fading frequency-nonselective channels. The decoder implements a combination of the Hungarian algorithm and Murty's algorithm for the kth assignment. Results show that the proposed algorithm achieves better coding gain when compared with hard-decision decoding. A salient feature of the decoder is that it operates within practical computational complexity, especially for large codebooks.**

***Index Terms*— Hungarian algorithm, Murty's algorithm, permutation codes, M-FSK, soft-decision decoding.**

I. Introduction

IN DESIGNING a decoding algorithm for forward error correction codes, coding gain and computational efficiency are major considerations in evaluating the performance of the decoding algorithm. It is therefore important for decoding algorithms to combine both low complexity and high coding gain performance in practical environments.

M-FSK modulation provides constant envelope modulation. Assuming equal energy signals are transmitted in the AWGN channel, Envelope Detection then becomes suitable to recover the message by using M correlators at the receiver [1].

Vinck [2] showed the effects of different noise conditions on M-FSK signals assuming a Power line Communication (PLC) Channel. Vinck *et al*. [3] further showed that permutation coding with M-FSK can handle narrow-band, impulse noise and also background noise within a certain distance of transmission.

When M-FSK is combined with permutation codes, additional frequencies help spread the information over additional time slots, which aids performance against impulse and narrow-band noise [4]. Shum [5] also showed that permutation block codes give better bit error rate performance when compared with convolutional codes soft-decoded with the Viterbi algorithm in some signal-to-noise ratio (SNR) regions.

With code construction examples in [6] and [7], existing decoding algorithms rely on the code construction method. We therefore introduce an efficient soft-decision decoder of permutation block codes irrespective of the code construction algorithm.

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We modulate with M-FSK and evaluate the performance of the proposed soft-decision decoder which implements the Hungarian Algorithm [8] for maximum assignment and Murty's algorithm for the k-th assignment [9]. The performance of the decoder is analyzed in the presence of AWGN and Rayleigh fading channel conditions.

II. M-FSK and Permutation Codes

Given a set of integers $\upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_L\}$, a permutation codebook C is defined as a subset of the set P containing all permutations of the integers υ, such that the minimum Hamming distance d_{min} of C is the largest, i.e., the minimum of the Hamming distances between any two permutations in C is optimized. The code rate of a permutation codebook is defined as [5]

$$R = \frac{\log_2(|C|)}{L\log_2(L)}, \tag{1}$$

where L is the length of each codeword and $|C|$ is the number of codewords in C.

We use a one-to-one mapping from the integers $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_L\}$ onto the $M = 2^m$ distinct frequencies of the M-FSK modulator, similar to [10] and [11], except now we use the integers of the codebook C, and not the elements from the field $GF(2^m)$. The vector representation of M-FSK of a transmitted signal for a sampling instance T, assuming f was transmitted can be generalized as

$$s_z(t) = \begin{cases} \sqrt{E_s}, & \text{if } f = f_z, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where E_s is the symbol energy and f_z is any of the M possible frequencies utilized to transmit any of the M possible signals $s_z(t)$ for $z = 1, 2, \ldots, M$.

III. Channel Model

At the output of the AWGN channel, a non-coherently received signal $r(t)$ of a transmitted signal $s_z(t)$ is represented as [1]

$$r(t) = s_z(t)e^{j\phi} + \eta(t), \quad \text{for } 0 \le t \le T, \tag{3}$$

where $\eta(t)$ is a random complex-valued white Gaussian process with zero-mean and variance of each complex component $\sigma^2 = 2N_0$.

The function of the non-coherent M-FSK detector is to choose the most likely transmitted frequency from a set of M frequencies, by choosing the one with the highest energy present at a sampling instance T [1]. The SNR for such a system is calculated as $SNR = E_s/N_0$ (refer to [1]). The non-coherent M-FSK detector consists of a bank of M pairs of quadrature correlators, one pair for each frequency to be detected.

The output of each quadrature pair is a metric, which is calculated using the square law. Each metric corresponds to each possible frequency. The most likely transmitted symbol for sampling instance T is determined based on these M metrics, by choosing the symbol corresponding to the metric with the highest value as the output of the envelope detector [1].

The effect of Rayleigh fading on the transmitted signal $s_z(t)$ is multiplicative [1]. The received lowpass signal is therefore

$$r(t) = \alpha e^{j\phi} s_z(t) + \eta(t), \quad \text{for } 0 \le t \le T, \tag{4}$$

where α is the fading constant and is modeled as a random variable with a complex-valued Gaussian distribution. The channel model assumes a slow fading frequency non-selective channel such that a fading condition persists for a length of the signaling time. We assume fading is slow enough to estimate ϕ and therefore achieve coherent detection [1].

IV. Soft-Decision Decoder

The soft-decision decoder aims to iteratively rank the costs of the input signal matrix until the assignment produces a codeword in C. In this section, we discuss the two algorithms combined in the decoder, the first determines the highest cost G_1 of the signal matrix which produces an equivalent codeword A_1. The second algorithm iteratively ranks the costs from the second-highest cost to the k-th highest cost $G_2, G_3, \ldots, G_k$ each producing an equivalent codeword $A_2, A_3, \ldots, A_k$.

A. Hungarian Algorithm

In linear programming, if the cost G_1 is such that we minimize

$$G_1 = \sum_{i=1}^{u} \sum_{j=1}^{u} h_{ij} x_{ij}, \tag{5}$$

subject to

$$\sum_{i=1}^{u} x_{ij} = 1, \quad (i = 1, \ldots, u),$$
$$\sum_{i=1}^{u} x_{ij} = 1, \quad (j = 1, \ldots, u), \tag{6}$$

where $x_{ij} \ge 0$, u is a positive integer and the $u \times u$ square cost matrix $H = [(h_{ij})]$, then G_1 is the cost of the assignment [9].

The Hungarian algorithm [8] therefore finds the permutation matrix $X = [(x_{ij})]$ that yields the lowest cost G_1, given a square cost matrix. X is such that only one element on each row and column is 1 while all other elements are 0. The assignment solution's row-column pair equivalent [9] is

$$a_k = \{(1, j_1), (2, j_2), \ldots, (u, j_u)\}, \tag{7}$$

where each pair represents the cell in X with value of 1 and $j_1, j_2, \ldots, j_u$ is a permutation of integers $1, 2, \ldots, u$. If $u = L$ and $j_1, j_2, \ldots, j_u$ is a permutation of integers in $\upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_L\}$, then each permutation codeword in C is a possible solution to the assignment problem. Each row-column pair represents a cell in the cost matrix and the sum of the cell values of all the row-column pairs in a_k produces the cost G_k.

Permutation block codes are constructed such that each symbol appears only once in the codeword. The Hungarian assignment solution (for minimum cost) also assigns each job to each worker such that a worker can only be assigned one task/job, the total sum of all costs gives the minimal cost of assignment. The cost matrix must be square, having the same number of jobs as workers.

A signal $r(t)$ at the input of the Hungarian algorithm decoder is negated before finding the minimum cost

$$r(t)' = -1 \times r(t). \tag{8}$$

As an example, a one-to-one mapping of randomly generated symbols to the cyclically rotated set of codewords $C_E = \{1234, 4123, 3412, 2341\}$ with $d_{min} = 4$ [5] produces a received R_f using (3) and negated signal R'_f using (8). Assuming $f = 1$, then

$$R'_1 = \begin{bmatrix} 0.1389 & \boxed{-0.3035} & 0.9411 & 0.0511 \\ \boxed{0.0985} & 0.1106 & -0.0253 & 1.1810 \\ 0.9716 & 0.0995 & 0.0668 & \boxed{-0.2412} \\ 0.0701 & 0.8874 & \boxed{-0.0914} & -0.0921 \end{bmatrix},$$

at the output of the channel. For each transmitted symbol, the received matrix R_f is obtained by mapping the transmitted symbol onto an L-tuple codeword in C. Using (2), modulation of each symbol then transforms the message to R_f, an $M \times L$ matrix.

The sum of the boxed values in R'_1 equals -0.5376 which is the minimum cost of R'_1 using [8]. The output codeword A_1 in this case is 2143 and its row-column representation is

$$a_1 = \{(1, 2), (2, 1), (3, 4), (4, 3)\}. \tag{9}$$

If $A_1 \in C_E$, the decoder stops and outputs A_1 as the likely transmitted codeword. However $A_1 \notin C_E$ implies an invalid codeword. The decoder therefore continues to Murty's algorithm to iteratively find $A_2, A_3, \ldots, A_k$.

B. Murty's Algorithm

Using the Hungarian algorithm's solution matrix $\bar{R}'_1$ that solved for A_1, Murty's algorithm [9] solves assignments $A_2, A_3, \ldots, A_k$. Nodes $N_1, N_2, \ldots, N_{u-1}$ are obtained by partitioning $\bar{R}'_1$ with a_1 such that

$$N_1 = \{\overline{(1, j_1)}\},$$
$$N_2 = \{(1, j_1), \overline{(2, j_2)}\},$$
$$\vdots$$
$$N_{u-1} = \{(1, j_1), (2, j_2), \ldots, \overline{(u-1, j_{u-1})}\}. \tag{10}$$

In each node, the row-column pair without the bar implies the values on the row and column are removed from $\bar{R}'_1$ for that node while the row-column pair with the bar implies the item at that row-column position is replaced with a large number. The minimum cost is then solved for each node. The node with the least cost forms the next assignment a_2. Since $\bar{R}'_1$ of A_1 is

$$\bar{R}'_1 = \begin{bmatrix} 0.3179 & \boxed{0} & 1.2439 & 0.3546 \\ \boxed{0} & 0.1366 & 0 & 1.2070 \\ 1.0883 & 0.3407 & 0.3073 & \boxed{0} \\ 0.0377 & 0.9795 & \boxed{0} & 0 \end{bmatrix},$$

partitioning $\bar{R}'_1$ with a_1 produces the nodes

$$N_1 = \begin{bmatrix} \boxed{0.3179} & 10^3 & 1.2439 & 0.3546 \\ 0 & \boxed{0.1366} & 0 & 1.2070 \\ 1.0883 & 0.3407 & 0.3073 & \boxed{0} \\ 0.0377 & 0.9795 & \boxed{0} & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 10^3 & \boxed{0} & 1.2070 \\ 1.0883 & 0.3073 & \boxed{0} \\ \boxed{0.0377} & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} \boxed{0.3073} & 10^3 \\ 0 & \boxed{0} \end{bmatrix}.$$

Since node N_2 has the lowest cost of 0.0377, the output of the second iteration A_2 in $\bar{R}'_1$ decodes as 4123 and

$$a_2 = \{(1,4), (2,1), (3,2), (4,3)\}. \tag{11}$$

Since $A_2 \in C_E$, the decoder stops and outputs A_2. For $A_2 \notin C_E$, a_2 is used to partition the new solution matrix of A_2 for the next assignment.

C. Decoder's Optimum Performance

In (5), if $h_{ij} = R'_f$ still subject to (6), then the decoder's optimum performance

$$G = \sum_{i=1}^{u} \sum_{j=1}^{u} R'_f x_{ij}, \tag{12}$$

where $u = L$ and $X = [(x_{ij})]$ is the permutation matrix of the codeword in C that minimizes G.

V. Simulation Results

Simulations are carried out by comparing the performance of the soft-decision decoder with hard-decision and optimum decoding. Hard-decision combines Envelope Detection with minimum distance decoding and the codebook is chosen such that $L = M$.

Tables I and II compare the performance of the soft-decision decoder over hard-decision for large codebooks in AWGN and AWGN plus Rayleigh fading channels respectively. The performance of the soft-decision decoder is also compared with the optimum performance in Figs. 1 - 3.

The value of the $\frac{|C|}{|P|}$ ratio affects the error-correction performance of the soft-decision decoder, especially at the first iteration. The lower the ratio, the higher the probability of outputting a codeword outside C. In the AWGN channel, this effect of values selected between $\frac{|C|}{|P|} = 0.1667$ and $\frac{|C|}{|P|} = 1$ is shown in Figs. 1 and 2.

The performance of the decoder also depends on d_{min} of the codebook. For example, the soft-decision decoder produces better performance for codebook $|C| = 12$ and $d_{min} = 2$ compared with a similar codebook, but with $d_{min} = 3$. When $\frac{|C|}{|P|} = 1$, every output of the decoder will always be a codeword in C at the first iteration. Therefore, computational performance of the decoder is minimal at this condition.

The performance of the soft-decision decoder with the addition of Rayleigh slow fading channel is shown in Fig. 3

TABLE I

PERFORMANCE OF SOFT-DECISION DECODER USING 8-FSK IN AWGN CHANNEL, NON-COHERENT DETECTION

				Gain (dB)					
$\|C\|$	R	$\frac{\|C\|}{\|P\|}$	d_{min}	A_1	A_2	A_3	A_4	A_5	A_6
8	0.125	0.0002	8	−0.3	−0.3	−0.2	−0.2	−0.2	−0.2
305	0.344	0.0076	5	0	0.1	0.2	0.4	0.5	0.5
1417	0.436	0.035	4	0	0.8	1.0	1.0	1.0	1.0
5000	0.512	0.124	3	0.6	1.2	1.3	1.3	1.3	1.3
10000	0.554	0.248	3	0.6	1.3	1.4	1.4	1.4	1.4
15000	0.578	0.372	3	0.6	1.2	1.3	1.3	1.3	1.3
20160	0.596	0.5	3	0.8	1.5	1.6	1.6	1.6	1.6
40320	0.637	1	2	2.3	2.3	2.3	2.3	2.3	2.3

TABLE II

PERFORMANCE OF SOFT-DECISION DECODER USING 8-FSK IN AWGN AND RAYLEIGH FADING CHANNELS

				Gain (dB)					
$\|C\|$	R	$\frac{\|C\|}{\|P\|}$	d_{min}	A_1	A_2	A_3	A_4	A_5	A_6
8	0.125	0.0002	8	−0.3	−0.3	−0.2	−0.2	−0.2	−0.2
305	0.344	0.0076	5	0	0.1	0.2	0.4	0.5	0.5
1417	0.436	0.035	4	0	0.8	1.0	1.0	1.0	1.0
5000	0.512	0.124	3	0.8	1.5	1.5	1.5	1.5	1.5
10000	0.554	0.248	3	1.0	1.5	1.5	1.5	1.5	1.5
15000	0.578	0.372	3	1.2	1.8	1.8	1.8	1.8	1.8
20160	0.596	0.5	3	1.8	1.8	1.8	1.8	1.8	1.8
40320	0.637	1	2	2.0	2.0	2.0	2.0	2.0	2.0

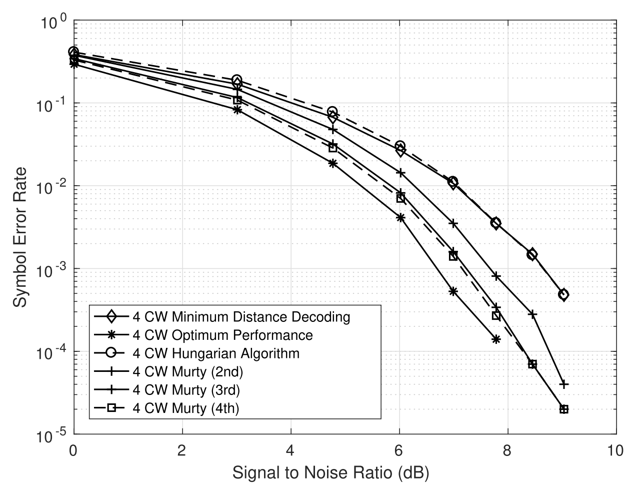

Fig. 1. Performance of soft-decision decoder in AWGN channel using 4 codewords (CW).

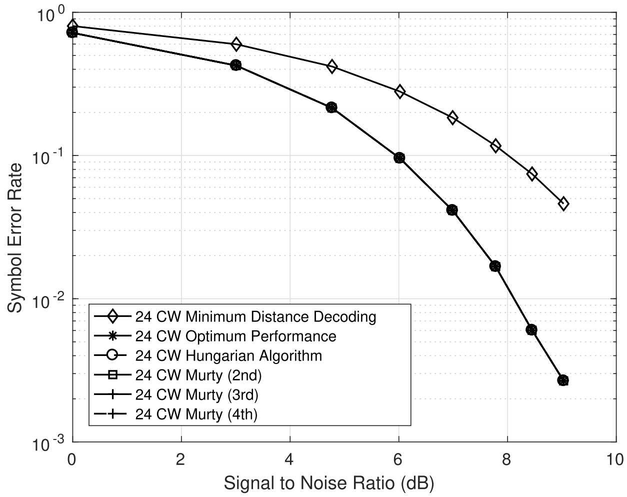

Fig. 2. Performance of soft-decision decoder in AWGN channel using 24 codewords (CW).

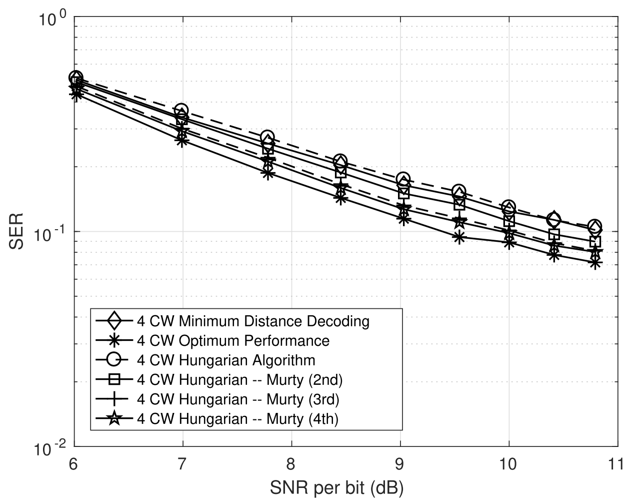

Fig. 3. Performance of soft-decision decoder in AWGN & rayleigh fading channels using 4 codewords (CW).

using 4 codewords while large size codebooks are shown in Table II. The behavior of the decoder relative to the different code rates remain similar to the AWGN channel.

The computational complexity of the soft-decision decoder becomes a major advantage over hard-decision for large size codebooks. For example, an 8-FSK system requires a codebook with $L = 8$ and $|P| = 40320$. Large codebooks constructed from $|P|$ means it will be computationally too complex to decode using a look-up table. However, because the input to the soft-decision decoder is always an $M \times M$ matrix, the same complexity is required by the decoder irrespective of the size of $|C|$. The worst case order of complexity of the decoder for an n-size input is $O(n^4)$ which combines the complexity of the Hungarian algorithm ($O(n^3)$) [12] and Murty's algorithm ($O(n^4)$) [13].

VI. Conclusion

We designed a soft-decision decoder to decode permutation block codes in AWGN and Rayleigh Fading Channels. The decoder combines the Hungarian algorithm for maximum assignment and Murty's algorithm for the k-th assignment. A positive observation is that the computational performance of the decoder is not exponential, yet applicable to code books of large sizes. Results compared the performance of the decoding algorithms with Envelope Detection plus Hard-decision decoding up to the fourth iteration for $L = 4$ and sixth iteration for $L = 8$.

The results show the performance of the Hungarian algorithm improved as the code rate increased. The performance of the Hungarian algorithm decoder is improved by iteratively using Murty's algorithm for the k-th assignment in order of decreasing costs. Murty's algorithm is only applied if the decoded codeword $A_1 \notin C$. With this condition, the complexity of the decoder is not always at its maximum.

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