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Stabilization of ODE with hyperbolic equation actuator subject to boundary control matched disturbance

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ABSTRACT
In this paper, we consider stabilisation for a cascade of ODE and first-order hyperbolic equation with external disturbance flowing to the control end. The active disturbance rejection control (ADRC) and sliding mode control (SMC) approaches are adopted in investigation. By ADRC approach, the disturbance is estimated through a disturbance estimator with both time-varying high gain and constant high gain, and the disturbance is canceled online in the feedback loop. It is shown that the resulting closed-loop system with time-varying high gain is asymptotically stable and is practically stable with constant high gain. By SMC approach, the existence and uniqueness of the solution for the closed loop via SMC are proved, and the monotonicity of the 'reaching condition' is presented. The resulting closed-loop system is shown to be exponentially stable. The numerical experiments are carried out to illustrate effectiveness of the proposed control law.

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1. Introduction

Stabilisation for systems described by partial differential equations (PDEs) is a key issue in distributed parameter control systems in the last several decades, see Chen, Delfour, Krall, and Payre (1987); Dos Santos, Bastin, Coron, and d'Andrea Novel (2008); Krstic and Smyshlyaev (2008); Komornik and Zuazua (1990); Luo, Guo, and Morgul (1999) and the references therein. For PDEs with boundary control matched disturbance, stability problems are recently studied in Cheng, Radisavljevic, and Su (2011); Ge, Zhang, and He (2011); Guo and Jin (2013a, 2013b); Guo and Liu (2014); Guo and Guo (2013); Guo, Zhou, Alfahaid, Younas, and Asiri (2014); Liu, Chen, and Wang (2016); Liu and Wang (2015); and Pisano, Orlov, and Usai (2011). Recently, stabilisation of a heat equation coupled Ordinary differential equation (ODE) system was considered in Wang, Liu, Ren, and Chen (2015), where the authors obtained the exponential stability by using sliding mode control (SMC). In this paper, we consider stability for an ODE system with one-dimensional hyperbolic equation actuator dynamics where the external disturbance flows to the boundary control end. To be motivated, let us consider a controlled ODE with disturbance described by

$$\dot{X} = AX(t) + B(U(t-1) + d(t-1)),\ t > 0, \tag{1.1}$$

where A is an $n \times n$ matrix, $B \in \mathbb{R}^{n\times 1}$, $U(t)$ and $d(t)$ are the control input and the external disturbance, both with a time delay normalised to be one. Let

$$z(x,t) = U(t+x-1) + d(t+x-1),\ x \in (0,1). \tag{1.2}$$

Then, $z(x, t)$ satisfies a one-dimensional hyperbolic PDE:

$$z_t(x,t) = z_x(x,t),\ x \in (0,1), t > 0. \tag{1.3}$$

So the control problem (1.1) can be reformulated as a coupled ODE-PDE system of the following:

$$\begin{cases} \dot{X} = AX(t) + Bz(0,t),\ t \geq 0, \\ z_t(x,t) = z_x(x,t),\ x \in (0,1),\ t > 0, \\ z(1,t) = U(t) + d(t),\ t \geq 0, \\ X(0) = X_0, \\ z(x,0) = z_0(x),\ 0 \leq x \leq 1, \end{cases} \tag{1.4}$$

where the PDE part is considered as controller and the original control plant ODE is connected with the PDE through boundary output of the PDE. It is seen that the time delay disappears in the new formulated system (1.4). This point of view clearly shows the infinite-dimensional nature of the delay systems.

In this paper, we consider stabilisation for a general cascade linear time invariant system and first-order

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hyperbolic partial integro-differential equation actuator dynamics described by

$$\begin{cases} \dot{X}(t) = AX(t) + Bw(0,t), \ t \geq 0, \\ w_t(x,t) = w_x(x,t) + g(x)w(0,t) \\ \quad + \int_0^x f(x,y)w(y,t)dy, \ x \in (0,1), \ t > 0, \\ w(1,t) = U(t) + d(t), \ t \geq 0, \\ X(0) = X_0, \\ w(x,0) = w_0(x), \ 0 \leq x \leq 1, \end{cases} \tag{1.5}$$

where $X \in \mathbb{R}^{n\times 1}$ and $w \in L^2(0,1)$ are the states of ODE and PDE, respectively, $U \in L^2_{loc}(0,\infty)$ is the control input of the entire system, A is an $n \times n$ matrix, $B \in \mathbb{R}^{n\times 1}$, and $d(t)$ is the external disturbance. It is supposed that the ODE part $\Sigma(A,B)$ is stabilisable, and the disturbance $d(t)$ is uniformly bounded, i.e. $|d(t)| \leq M$ for some $M > 0$ and all $t \geq 0$. We will drop the obvious domains for both time and spatial variables throughout the paper.

The objective of the present paper is to design feedback controls that can stabilise system (1.5) depicted in Figure 1 in the state space $\mathbb{H} = \mathbb{R}^n \times L^2(0,1)$ by rejecting disturbance. First, the inner product of $\mathbb{H}$ is defined by

$$\langle (X_1,h_1),(X_2,h_2)\rangle = X_1^\top \overline{X_2} + \int_0^1 h_1(x)\overline{h_2(x)}dx, \quad \forall\ (X_i,h_i) \in \mathbb{H}, i = 1,2. \tag{1.6}$$

We proceed as follows. In Section 2, we first transform the original system into an equivalent system by backstepping method. The well-posedness of open-loop system is discussed. In Section 3, we first design a time-varying disturbance estimator and show the asymptotic stability of the resulting closed loop and then a constant high-gain disturbance estimator and show the practical stability of its corresponding closed-loop system. Section 4 is devoted to disturbance rejection by the SMC, for which the existence and uniqueness of solution are proved, and the monotonicity of the 'reaching condition' is presented. Some numerical simulations are presented for illustration in Section 5.

2. Backstepping design and well-posedness

First of all, for system (1.5), if $d(t)$ is absent, then the system is stabilisable and the state feedback control can be obtained by the backstepping method as Krstic (2009, Theorem 14.7),

$$U(t) = Ke^A X(t) + \int_0^1 k(1,y)w(y,t)dy, \tag{2.1}$$

where K is chosen so that $A + BK$ is Hurwitz, and $k(x,y)$ satisfies

$$\begin{cases} k_x(x,y) + k_y(x,y) = \int_y^x k(x,\xi)f(\xi,y)d\xi - f(x,y), \\ k(x,0) = \int_0^x k(x,y)g(y)dy - g(x) + Ke^{Ax}B. \end{cases} \tag{2.2}$$

The existence of solution to the kernel Equation (2.2) can be found in Krstic (2009, Theorem 14.6).

Following an idea of Krstic (2009), we introduce an invertible transformation:

$$\begin{cases} X(t) = X(t), \\ v(x,t) = w(x,t) \\ \quad - \int_0^x k(x,y)w(y,t)dy - Ke^{Ax}X(t), \end{cases} \tag{2.3}$$

under which, the original system (1.5) is transformed into system of the following:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bv(0,t), \\ v_t(x,t) = v_x(x,t), \ x \in (0,1), \\ v(1,t) = U(t) + d(t) \\ \quad - \int_0^1 k(1,y)w(y,t)dy - Ke^A X(t), \\ X(0) = X_0, \\ v(x,0) = v_0(x). \end{cases} \tag{2.4}$$

Introduce a new control variable $U_0(t)$ by designing

$$U(t) = U_0(t) + \int_0^1 k(1,y)w(y,t)dy + Ke^A X(t). \tag{2.5}$$

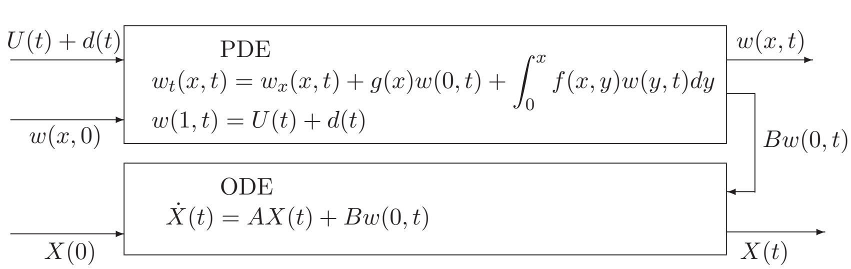


Figure 1. Block diagram of coupled PDE-ODE system (1.5).

Then system (2.4) becomes

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bv(0,t), \\ v_t(x,t) = v_x(x,t), \\ v(1,t) = U_0(t) + d(t), \\ X(0) = X_0, \\ v(x,0) = v_0(x). \end{cases} \tag{2.6}$$

We write system (2.6) in operator form:

$$\frac{d}{dt}Z(\cdot,t) = \mathbb{A}Z(\cdot,t) + \mathbb{B}(U_0(t) + d(t)), \tag{2.7}$$

where $Z(\cdot,t) = [X(t), v(\cdot,t)]^\top$, $\mathbb{B} = [0, \delta(x-1)]^\top$, and $\mathbb{A}$ is a linear operator defined in $\mathbb{R}^n \times L^2(0,1)$ by

$$\mathbb{A}[X,h]^\top = [(A+BK)X + Bh(0), h']^\top, \quad D(\mathbb{A}) = \{[X,h]^\top \in \mathbb{R}^n \times H^1(0,1) : h(1) = 0\}. \tag{2.8}$$

A direct computation shows that $\mathbb{A}^*$, the adjoint operator of $\mathbb{A}$, is given by

$$\mathbb{A}^*[X,h]^\top = [(A+BK)^\top X, -h']^\top, \quad D(\mathbb{A}^*) = \{[X,h]^\top \in \mathbb{R}^n \times H^1(0,1) : B^\top X = h(0)\}, \tag{2.9}$$

and $\mathbb{B}^*$, the adjoint operator of $\mathbb{B}$, is given by

$$\mathbb{B}^*[X,h]^\top = h(1), \ \ D(\mathbb{B}^*) = \mathbb{R}^n \times H^1(0,1). \tag{2.10}$$

Lemma 2.1: *Let $\mathbb{A}$ be defined by (2.8). Then $\mathbb{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathbb{A})$, the spectrum of $\mathbb{A}$, consists of isolated eigenvalues of finitely algebraic multiplicity only. Moreover, $\sigma(\mathbb{A}) = \sigma(A+BK)$.*

Proof: For any given $[X_1, h_1]^\top \in \mathbb{H}$, solve

$$\mathbb{A}[X,h]^\top = [(A+BK)X + Bh(0), h']^\top = [X_1, h_1]^\top \tag{2.11}$$

to obtain

$$X = (A+BK)^{-1}\left(X_1 + B\int_0^1 h_1(s)ds\right), \quad h(x) = -\int_x^1 h_1(s)ds. \tag{2.12}$$

Hence, $\mathbb{A}^{-1}$ exists and is compact on $\mathbb{H}$ by the Sobolev embedding theorem (Adams & Fournier, 2003, p. 97). Therefore, $\sigma(\mathbb{A})$ consists of isolated eigenvalues of finitely algebraic multiplicity only (Müller, 2007, Theorem 13, p. 154).

Next, we show $\sigma(\mathbb{A}) = \sigma(A+BK)$. Actually, for any $\lambda_0 \in \sigma(A+BK)$, let X_0 be an eigenvector corresponding to λ_0. We can see that $\lambda_0 \in \sigma(\mathbb{A})$ and a corresponding eigenfunction is given by $[X_0, 0]^\top$. On the other hand, for any $\lambda \notin \sigma(A+BK)$, $(\lambda I_{n\times n} - (A+BK))^{-1}$ exists, where $I_{n\times n}$ is the $n \times n$ identity matrix. For any given $[X_1, h_1]^\top \in \mathbb{H}$, solve

$$(\lambda I - \mathbb{A})[X,h]^\top = [\lambda X - (A+BK)X - Bh(0), \lambda h - h']^\top = [X_1, h_1]^\top \tag{2.13}$$

to obtain

$$X = (\lambda I_{n\times n} - (A+BK))^{-1}\left(X_1 + \int_0^1 e^{-\lambda s}h_1(s)ds\right), \quad h(x) = \int_x^1 e^{\lambda(x-s)}h_1(s)ds, \tag{2.14}$$

which means $\lambda \in \rho(\mathbb{A})$ if $\lambda \notin \sigma(A+BK)$. Therefore, $\sigma(\mathbb{A}) = \sigma(A+BK)$. ■

Lemma 2.2: *The operator $\mathbb{A}$ defined by (2.8) generates an exponential stable C_0-semigroup $e^{\mathbb{A}t}$ on $\mathbb{H}$ and $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$. Therefore, for any initial value $[X(0), v(\cdot,0)]^\top \in \mathbb{H}$ and control input $U_0 \in L^2_{loc}(0,\infty)$ and $d \in L^2_{loc}(0,\infty)$, (2.7) admits a unique solution $[X(t), v(\cdot,t)]^\top \in C(0,+\infty;\mathbb{H})$.*

Proof: Since $A+BK$ is Hurwitz, there exists a positive definite matrix P such that

$$P(A+BK) + (A+BK)^\top P = -I_{n\times n}.$$

Define a new inner product in $\mathbb{H}$ as follows:

$$\langle [X_1,h_1]^\top, [X_2,h_2]^\top \rangle_1 = X_1^\top P\overline{X_2} + \iota\int_0^1 e^x h_1(x)\overline{h_2(x)}dx, \ \ \forall\ [X_i,h_i]^\top \in \mathbb{H}, \ \ i = 1,2. \tag{2.15}$$

where $\iota > 2\|B^\top P\|^2/e$. It is obvious that the norm induced by (2.15) is equivalent to the one induced by (1.6). Under the new inner product, for any $Z = [X,h]^\top \in D(\mathbb{A})$,

$$\begin{aligned}\langle \mathbb{A}Z, Z\rangle_1 &= \langle [(A+BK)^\top X + B^\top h(0), h']^\top, [X,h]^\top\rangle_1 \\ &= [X^\top(A+BK)^\top + B^\top h(0)]P\overline{X} \\ &\quad + \iota\int_0^1 e^x h'(x)\overline{h(x)}dx,\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Re}\langle \mathbb{A}Z, Z\rangle_1 &= \frac{1}{2}\mathrm{Re}\left(X^\top(A+BK)^\top P\overline{X} + X^\top P(A+BK)\overline{X}\right) \\ &\quad + \mathrm{Re}(h(0)B^\top P\overline{X})\end{aligned}$$

$$
\begin{aligned}
&+\frac{\iota}{2}e^{1}|h(1)|^2-\frac{\iota}{2}e^{1}|h(0)|^2-\frac{\iota}{2}\int_0^1 e^x|h(x)|^2dx\\
&\le -\frac{1}{2}|X|^2-\frac{\iota}{2}\int_0^1 e^x|h(x)|^2dx+\frac{1}{4}|X|^2\\
&\quad-\left(\frac{\iota}{2}e-\|B^\top P\|^2\right)|h(0)|^2\\
&\le -\min\left\{\frac{1}{4\lambda_{\max}(P)},\frac{1}{2}\right\}\|[X,h]^\top\|_1^2,
\end{aligned}
$$

where $\lambda_{\max}(P)$ is the maximal eigenvalue of positive matrix P. Hence, $\mathbb{A}+\min\{1/(4\lambda_{\max}(P)),1/2\}I$ is dissipative and so is for $\mathbb{A}$. By Lemma 2.1 and the Lumer-Phillips theorem (Pazy, 1983, Theorem 1.4.3), $\mathbb{A}$ generates a C_0-semigroup of contractions on $\mathbb{H}$, and so does for $\mathbb{A}+\min\{1/(4\lambda_{\max}(P)),1/2\}I$. Therefore, the semigroup generated by $\mathbb{A}$ is exponentially stable and its decay rate is given by $-\min\{1/(4\lambda_{\max}(P)),1/2\}$.

Now we show that $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$ (Weiss, 1989). Actually, a straightforward computation gives

$$
\begin{aligned}
\mathbb{A}^{*-1}[X,h]^\top &= [(A+BK)^{\top-1}X, B^\top(A+BK)^{\top-1}X\\
&\quad-\int_0^x h(s)ds]^\top,
\end{aligned}
$$

and

$$
\mathbb{B}^*\mathbb{A}^{*-1}[X,h]^\top = B^\top(A+BK)^{\top-1}X-\int_0^1 h(s)ds, \tag{2.16}
$$

which is bounded from $\mathbb{H}$ to $\mathbb{C}$. Now, we consider the dual system associated with $\mathbb{A}^*$ as follows:

$$
\frac{d}{dt}Z^*(\cdot,t)=\mathbb{A}^*Z^*(\cdot,t),\ \ y(t)=\mathbb{B}^*Z^*(\cdot,t), \tag{2.17}
$$

that is,

$$
\begin{cases}
\dot{X}^*(t)=(A+BK)^\top X^*,\\
w_t^*(x,t)=-w_x^*(x,t),\\
w^*(0,t)=B^\top X^*(t),\ \ t\ge 0,\\
X^*(0)=X_0^*,\ \ w^*(x,0)=w_0^*(x),\\
y(t)=w^*(1,t).
\end{cases} \tag{2.18}
$$

It is seen that the solution $X^*(t)$ of 'ODE part' of (2.18) is exponentially stable, that is,

$$
\begin{aligned}
X^*(t)&=e^{(A+BK)^\top}X^*(0),\ \ |X^*(t)|\\
&\le M_0e^{-\omega t}|X_0^*| \text{ for some } M_0,\omega>0 \text{ and all } t\ge 0.
\end{aligned} \tag{2.19}
$$

Define the energy function for (2.18) as

$$
E(t)=X^{*\top}(t)PX^*(t)+\int_0^1 (w^*(x,t))^2dx.
$$

Differentiating $E(t)$ with respect to t along the solution to (2.18) and using (2.19) yields

$$
\begin{aligned}
&\dot{E}(t)\\
&=-2|X^*(t)|^2-\frac{1}{2}(w^*(1,t))^2+\frac{1}{2}(w^*(0,t))^2\\
&\le\left(\frac{\|B\|}{2}-2\right)|X^*(t)|^2\\
&\le M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}|X^*(0)|^2.
\end{aligned} \tag{2.20}
$$

By Taylor's formula and (2.20), for any $T\ge 0$,

$$
\begin{aligned}
&E(T)\\
&\le E(0)+M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}|X^*(0)|^2T\\
&\le\left(M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}\frac{1}{\lambda_{\min}(P)}T+1\right)E(0),
\end{aligned} \tag{2.21}
$$

and

$$
\begin{aligned}
&\int_0^T E(s)ds\\
&\le\left(M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}\frac{1}{2\lambda_{\min}(P)}T^2+T\right)E(0),
\end{aligned} \tag{2.22}
$$

where $\lambda_{\min}(P)$ is the minimal eigenvalue of positive matrix P. Let

$$
\varrho(t)=\int_0^1 e^x(w^*(x,t))^2dx.
$$

Differentiate $\varrho(t)$ to give

$$
\begin{aligned}
\frac{e}{2}\int_0^T(w^*(1,t))^2dt &\le \varrho(0)-\varrho(T)+\frac{(w^*(0,t))^2}{2}\\
&\quad+\frac{e}{2}\int_0^T E(s)ds.
\end{aligned} \tag{2.23}
$$

Since $\varrho(t)\le 2eE(t)$, it follows from (2.19), (2.21), (2.22), and (2.23) that

$$
\begin{aligned}
&\int_0^T y^2(s)ds\\
&\le\frac{2}{e}\left[2eE(0)+2eE(T)+\frac{\|B\|^2}{2}M_0^2|X^*(0)|^2\right]
\end{aligned}
$$

$$+\int_0^T E(s)ds$$
$$\leq\left[4M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}\frac{1}{\lambda_{\min}(P)}T+8\right.$$
$$\left.+\frac{\|B\|^2}{e}M_0^2\right]E(0)$$
$$+\left(M_0^2\max\left\{\left(\frac{\|B\|}{2}-2\right),0\right\}\frac{1}{2\lambda_{\min}(P)}T^2+T\right)E(0).$$

This fact, together with the boundedness of $\mathbb{B}^*\mathbb{A}^{*-1}$, shows the admissibility of $\mathbb{B}$ (Weiss, 1989). By Tucsnak and Weiss (2009, Proposition 4.2.5), for any initial value $[X(0),v(\cdot,0)]^\top\in\mathbb{H}$, control input $U_0\in L^2_{loc}(0,\infty)$, and $d\in L^2_{loc}(0,\infty)$, (2.7) admits a unique solution $[X(t),v(\cdot,t)]^\top\in C(0,+\infty;\mathbb{H})$. ■

3. Active disturbance rejection control approach

In this section, we apply the active disturbance rejection control (ADRC) to estimate the disturbance in real time, which is the key feature of ADRC. To estimate the disturbance, we need to transform the disturbance from PDE to an ODE by special test function. By Lemma 2.2, the (weak) solution of (2.7) is understood in the sense of

$$\frac{d}{dt}\langle[X,v]^\top,[Y,h]^\top\rangle_{\mathbb{H}}=\langle[X,v]^\top,\mathbb{A}^*[Y,h]^\top\rangle_{\mathbb{H}}$$
$$+(U_0(t)+d(t))\mathbb{B}^*[Y,h]^\top,\ \forall\ [Y,h]^\top\in D(\mathbb{A}^*),\quad(3.1)$$

where $\mathbb{A}^*$ is given by (2.9) and $\mathbb{B}^*$ is given by (2.10).

Equation (3.1) demonstrates clearly the infinite-dimensional nature of PDE, and PDE (2.7) is equivalent to a system of infinitely many ODEs (3.1), where $[Y,h]^\top$ is called (smooth) test function. Notice that the disturbance $d(t)$ appears in ODEs (3.1). We will choose time-varying gain estimator and constant high gain to estimate the disturbance.

3.1 Disturbance estimator with a time-varying high gain

First, choose the time-dependent test function of the following:

$$[Y^t,h^t]^\top=[0,x\kappa(t)]^\top\in D(\mathbb{A}^*),\qquad(3.2)$$

where the positive differentiable continuous function $\kappa(t)$ is assumed to satisfy

$$\lim_{t\to\infty}\kappa(t)=0\text{ and }\left|\frac{\kappa'(t)}{\kappa(t)}\right|\text{ is bounded for all }t\geq 0.\qquad(3.3)$$

The function $\kappa(t)=\frac{1}{1+t}$ is one candidate satisfying condition (3.3). Substitute $[Y^t,h^t]^\top$ into (3.1) to obtain

$$\dot{y}_1(t)=y_2(t)+U_0(t)\kappa(t)+d(t)\kappa(t),\qquad(3.4)$$

where

$$y_1(t)=\int_0^1 v(x,t)x\kappa(t)dx,\ \ y_2(t)=\int_0^1 v(x,t)x\dot{\kappa}(t)dx$$
$$-\int_0^1 v(x,t)\kappa(t)dx.\qquad(3.5)$$

It is seen that (3.4) is an ODE with control $U_0(t)$ and disturbance $d(t)$ for which we can apply ADRC approach for lumped parameter systems to estimate the disturbance (Guo & Zhao, 2011). To this purpose, we design a time-varying, high-gain extended state observer for system (3.4) as follows (Guo et al., 2014):

$$\begin{cases}\dot{\hat{y}}_1(t)=y_2(t)+U_0(t)\kappa(t)+\hat{d}(t)\kappa(t)\\ \quad -r(t)[\hat{y}_1(t)-y_1(t)],\\ \dfrac{d}{dt}(\hat{d}(t)\kappa(t))=-r^2(t)[\hat{y}_1(t)-y_1(t)],\end{cases}\qquad(3.6)$$

where $r(t)$ is the time-varying gain and satisfies

$$\dot{r}(t)>0,\ \lim_{t\to\infty}r(t)=\infty,\ \frac{\dot{r}(t)}{r(t)}$$
$$\leq\overline{M},\ \forall\ t\geq 0\text{ for some }\overline{M}>0.\ (3.7)$$

Function $r(t)=1+t$ is one candidate satisfying condition (3.7). The system (3.6) is served as an state feedback disturbance estimator for system (2.6).

Lemma 3.1 (Guo et al., 2014)**:** *Assume that conditions (3.7) and (3.3) are satisfied and that the disturbance $d(t)$ is uniformly bounded and satisfies*

$$\lim_{t\to\infty}\frac{|\dot{d}(t)|}{r(t)}=0.\qquad(3.8)$$

Let $y_1(t)$ and $y_2(t)$ be defined by (3.5). Then the solutions of (3.6) satisfy

$$\lim_{t\to\infty}|\hat{y}_1(t)-y_1(t)|=\lim_{t\to\infty}|\hat{d}(t)-d(t)|=0.\ (3.9)$$

Now that we have an online estimation $\hat{d}(t)$ of the disturbance, the next step is naturally to compensate (cancel) the disturbance in the feedback loop since the operator $\mathbb{A}$ given by (2.8) generates actually an exponential stable C_0-semigroup on $\mathbb{H}$ by Lemma 2.2. This is done by designing

$$U_0(t)=-\hat{d}(t).\qquad(3.10)$$

It is seen that the control (3.10) is used to cancel the effect of disturbance. This is just the estimation/cancellation nature of ADRC approach. Under feedback (3.10), the closed loop of system (1.5) becomes

$$\begin{cases} \dot{X}(t) = AX(t) + Bw(0,t), \ t \geq 0, \\ w_t(x,t) = w_x(x,t) + g(x)w(0,t) \\ \quad + \int_0^x f(x,y)w(y,t)dy, \ x \in (0,1), \ t > 0, \\ w(1,t) = -\hat{d}(t) + d(t) \\ \quad + \int_0^1 k(1,y)w(y,t)dy + Ke^A X(t), \\ \dot{\hat{y}}_1(t) = y_2(t) + U_0(t)\kappa(t) \\ \quad + \hat{d}(t)\kappa(t) - r(t)[\hat{y}_1(t) - y_1(t)], \ t \geq 0, \\ \frac{d}{dt}\left(\hat{d}(t)\kappa(t)\right) = -r^2(t)[\hat{y}_1(t) - y_1(t)], \\ \quad t \geq 0, X(0) = X_0, \\ w(x,0) = w_0(x), \ x \in [0,1], \end{cases} \tag{3.11}$$

where $r(t)$ and $\kappa(t)$ are defined by (3.7) and (3.3), respectively.

Theorem 3.1: *Assume that conditions (3.7) and (3.3) are satisfied and that the disturbance $d(t)$ is uniformly bounded and satisfies (3.8). Let $y_1(t)$ and $y_2(t)$ be defined by (3.5). Then system (3.11) is asymptotically stable in the sense of*

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} \left\{ |X(t)|^2 + \int_0^1 w^2(x,t)dx + |\hat{y}_1(t)| + |\hat{d}(t) - d(t)| \right\} = 0. \tag{3.12}$$

Proof: Introduce the error variables $\tilde{y}_1(t) = r(t)[\hat{y}_1(t) - y(t)]$, $\tilde{d}(t) = \hat{d}(t) - d(t)$. Then under transformation (2.3), system (3.11) is equivalent to

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bv(0,t), \\ v_t(x,t) = v_x(x,t), \\ v(1,t) = -\tilde{d}(t), \\ \dot{\tilde{y}}_1(t) = -r(t)\tilde{y}_1(t) + r(t)\tilde{d}(t)\kappa(t) + \frac{\dot{r}(t)}{r(t)}\tilde{y}_1(t), \\ \frac{d}{dt}(\tilde{d}(t)\kappa(t)) = -r(t)\tilde{y}_1(t) - \frac{d}{dt}(d(t)\kappa(t)). \end{cases} \tag{3.13}$$

The convergence of $\tilde{d}(t) = \hat{d}(t) - d(t)$ to zero as $t \to \infty$ in (3.13) has been proven in (3.9). We need only show the convergence of the '(X, v) part' of (3.13), that is,

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \left[|X(t)|^2 + \int_0^1 v^2(x,t)dx \right] = 0. \tag{3.14}$$

This is because if (3.14) is true, then it follows from (3.9) that

$$\hat{y}_1(t) = \frac{\tilde{y}_1(t)}{r(t)} + y_1(t) = \frac{\tilde{y}_1(t)}{r(t)} + \kappa(t)\int_0^1 v(x,t)xdx \to 0 \text{ as } t \to \infty.$$

Now, we show (3.14). Similar to (2.7), we can write the '(X, v) part' of (3.13) as

$$\frac{d}{dt}\begin{pmatrix} X(t) \\ v(\cdot,t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} X(t) \\ v(\cdot,t) \end{pmatrix} + \mathbb{B}(-\tilde{d}(t)), \tag{3.15}$$

where $\mathbb{A}$ is given by (2.8) and $\mathbb{B} = [0, \delta(x-1)]^\top$. However, (3.15) is exactly the same as those in Guo et al. (2014) and Guo and Zhou (2014, 2015). Since $\mathbb{A}$ generates an exponential stable C_0-semigroup $e^{\mathbb{A}t}$ on $\mathbb{H}$ and $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$ by Lemma 2.2, and $|\tilde{d}(t)| \to 0$ as $t \to \infty$ by (3.9), we end the proof. ■

3.2 Disturbance estimator with a constant high gain

Choose the test functions of the following:

$$[Y, h]^\top = [0, -x]^\top \in D(\mathbb{A}^*). \tag{3.16}$$

Substitute $[0, -x]^\top$ into (3.1) to obtain

$$\dot{y}_1(t) = y_2(t) - (U_0(t) + d(t)), \tag{3.17}$$

where

$$y_1(t) = -\int_0^1 v(x,t)xdx, \ y_2(t) = \int_0^1 v(x,t)dx. \tag{3.18}$$

From this, we design a high gain disturbance estimator for system (3.17) as follows (Guo & Jin, 2013a):

$$\begin{cases} \dot{\hat{y}}_{1\varepsilon}(t) = y_2(t) - (U_0(t) + \hat{d}_\varepsilon(t)) - \frac{1}{\varepsilon}[\hat{y}_{1\varepsilon}(t) - y_1(t)], \\ \dot{\hat{d}}_\varepsilon(t) = -\frac{1}{\varepsilon^2}[\hat{y}_{1\varepsilon}(t) - y_1(t)], \end{cases} \tag{3.19}$$

where ϵ is the tuning small parameter and $\hat{d}_\varepsilon(t)$ is regarded as an approximation of $d(t)$, which is confirmed by Lemma 3.2.

Lemma 3.2 (Guo & Jin, 2013a; Guo, Liu, Alfahaid, Younas, & Asiri, 2015)**:** *Assume that the disturbance $d(t)$ and its derivative $\dot{d}(t)$ are uniformly bounded with up bound M. Let $y_1(t)$ and $y_2(t)$ be defined by (3.18). Then* *the solutions of* (3.19) *satisfy*

$$\lim_{t\to\infty} |\hat{y}_{1\varepsilon}(t) - y_1(t)| = \lim_{t\to\infty} |\hat{d}_\varepsilon(t) - d(t)| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0. \tag{3.20}$$

Moreover, for any fixed $T > 0$,

$$\begin{aligned}\int_0^T |\hat{y}_{1\varepsilon}(t) - y_1(t)|dt &= \int_0^T |\hat{d}_\varepsilon(t) - d(t)|dt \\ &= \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0,\\ \int_0^T |\hat{y}_{1\varepsilon}(t) - y_1(t)|^2dt &= \int_0^T |\hat{d}_\varepsilon(t) - d(t)|^2dt \\ &= \mathcal{O}(\varepsilon^{-1}) \text{ as } \varepsilon \to 0.\end{aligned} \tag{3.21}$$

By (3.21), $\int_0^T |\hat{d}_\varepsilon(t) - d(t)|dt$ is uniformly bounded in ϵ for any fixed $T > 0$, and $\int_0^T |\hat{d}_\varepsilon(t) - d(t)|^2dt$ is unbounded in ϵ. But from Lemma 2.2 and Weiss (1989, Theorem 4.8), we only have admissibility with $L^2_{\text{loc}}(0,\infty)$ control yet not the admissibility with $L^1_{\text{loc}}(0,\infty)$ control. This fact will be used in the proof of Theorem 3.2 later. To overcome this difficulty, we design the control as follows:

$$U_0(t) = -\text{sat}(\hat{d}_\varepsilon(t)), \tag{3.22}$$

where $\text{sat}(x) = \min\{M+1, \max\{x, -M-1\}\}$ is a saturate function. Under feedback (3.22), the closed loop of system (1.5) becomes

$$\begin{cases}\dot{X} = AX(t) + Bw(0,t),\\ w_t(x,t) = w_x(x,t) + g(x)w(0,t)\\ \quad + \int_0^x f(x,y)w(y,t)dy,\\ w(1,t) = -\text{sat}(\hat{d}_\varepsilon(t)) + d(t)\\ \quad + \int_0^1 k(1,y)w(y,t)dy + Ke^{A}X(t),\\ \dot{\hat{y}}_{1\varepsilon}(t) = y_2(t) - (-\text{sat}(\hat{d}_\varepsilon(t)) + \hat{d}_\varepsilon(t))\\ \quad - \frac{1}{\varepsilon}[\hat{y}_1(t) - y_1(t)],\\ \dot{\hat{d}}_\varepsilon(t) = -\frac{1}{\varepsilon^2}[\hat{y}_1(t) - y_1(t)],\\ X(0) = X_0,\\ w(x,0) = w_0(x).\end{cases} \tag{3.23}$$

Theorem 3.2: *Assume that the disturbance* $d(t)$ *and its derivative* $\dot{d}(t)$ *are uniformly bounded with up bound* M. *Let* $y_1(t)$ *and* $y_2(t)$ *be defined by* (3.18). *Then system* (3.23) *is practically stable in the sense of*

$$\limsup_{t\to\infty} E_\varepsilon(t) = \limsup_{t\to\infty} \left\{ |X(t)|^2 + \int_0^1 w^2(x,t)dx + |\hat{y}_{1\varepsilon}(t)| + |\hat{d}_\varepsilon(t) - d(t)| \right\} \le C\varepsilon, \tag{3.24}$$

where $C > 0$ *is a constant independent of* ϵ.

Proof: Introduce the error variables $\tilde{y}_{1\varepsilon}(t) = \hat{y}_{1\varepsilon}(t) - y_1(t)$, $\tilde{d}_\varepsilon(t) = \hat{d}_\varepsilon(t) - d(t)$. Then under transformation (2.3), system (3.23) is equivalent to

$$\begin{cases}\dot{X} = (A+BK)X(t) + Bv(0,t),\\ v_t(x,t) = v_x(x,t),\\ v(1,t) = -\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t),\\ \dot{\tilde{y}}_{1\varepsilon}(t) = -\frac{1}{\varepsilon}\tilde{y}_{1\varepsilon}(t) - \tilde{d}_\varepsilon(t),\\ \frac{d}{dt}\tilde{d}_\varepsilon(t) = \frac{1}{\varepsilon^2}\tilde{y}_{1\varepsilon}(t) - \dot{d}(t).\end{cases} \tag{3.25}$$

The convergence of $\tilde{d}_\varepsilon(t) = \hat{d}_\varepsilon(t) - d(t)$ in (3.25) has been proven in (3.20). We need to show only the convergence of the '(X, v) part' of (3.13), that is,

$$\limsup_{t\to\infty} F_\varepsilon(t) = \limsup_{t\to\infty}\left[|X(t)|^2 + \int_0^1 v^2(x,t)dx\right] \le C\varepsilon, \tag{3.26}$$

where $C > 0$ is a constant independent of ϵ. This is because if (3.26) is true, then it follows from (3.20) that

$$\begin{aligned}\limsup_{t\to\infty} |\hat{y}_{1\varepsilon}(t)| &= \limsup_{t\to\infty} |\tilde{y}_{1\varepsilon}(t) + y_1(t)|\\ &\le \limsup_{t\to\infty} |\tilde{y}_{1\varepsilon}(t)| + \limsup_{t\to\infty}\left|\int_0^1 v(x,t)xdx\right| \le C\varepsilon.\end{aligned}$$

Now, we show (3.26). Similar to (2.7), we can write the '(X, v) part' of (3.13) as

$$\frac{d}{dt}\begin{pmatrix} X(t)\\ v(\cdot,t)\end{pmatrix} = \mathbb{A}\begin{pmatrix} X(t)\\ v(\cdot,t)\end{pmatrix} + \mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t)), \tag{3.27}$$

where $\mathbb{A}$ is given by (2.8) and $\mathbb{B} = [0, \delta(x-1)]^\top$. Since for any given $\epsilon > 0(<1)$, by (3.20), there exist $T_0 > 0$ and $C_1 > 0$ such that $|-\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t)| < C_1\varepsilon$ for all $t \ge T_0$. Since $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$, we have

$$\begin{aligned}&\left\|\int_0^{T_0} e^{\mathbb{A}(T_0-s)}\mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s))ds\right\|\\ &\le C_{T_0}\|-\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)\|_{L^2(0,T_0)}\\ &\le C_{T_0}T_0(2M+1)^2\end{aligned}$$

for some C_{T_0} that is independent of $\tilde{d}_\varepsilon(s)$ and d. In above inequality, we have used the boundedness of saturation function, because without using the saturation function, we cannot get the boundedness of $\|-(\tilde{d}_\varepsilon(s) + d(s)) + d(s)\|_{L^2(0,T_0)}$ due to (3.21). On the other hand, since $e^{\mathbb{A}t}$

is exponentially stable, and $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$ with $L^2_{\text{loc}}(0,\infty)$ control and hence is admissible to $e^{\mathbb{A}t}$ with $L^\infty_{loc}(0,\infty)$ control, by Weiss (1989, Proposition 2.5), we have

$$\begin{aligned}&\left\| \int_{T_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))ds\right\|\\ &\quad= \left\| \int_0^t e^{\mathbb{A}(T_0-s)}\mathbb{B}(0\underset{T_0}{\diamondsuit}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))ds\right\|\\ &\quad\le L\|(0\underset{T_0}{\diamondsuit}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))\|_{L^\infty(0,\infty)}\le LC_1\varepsilon,\end{aligned}$$

where L is a constant that is independent of $u(s)$, and

$$(u\underset{\tau}{\diamondsuit}v)(t)=\begin{cases}u(t),\ 0\le t\le\tau,\\ v(t),\ t>\tau.\end{cases}$$

Noting that the solution is given by

$$\begin{aligned}\begin{pmatrix}X(t)\\ v(\cdot,t)\end{pmatrix}&=e^{\mathbb{A}t}\begin{pmatrix}X(0)\\ v(\cdot,0)\end{pmatrix}\\ &\quad+\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))ds\\ &=e^{\mathbb{A}t}\begin{pmatrix}X(0)\\ v(\cdot,0)\end{pmatrix}+e^{\mathbb{A}(t-T_0)}\\ &\quad\times\int_0^{T_0} e^{\mathbb{A}(T_0-s)}\mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))ds\\ &\quad+\int_{T_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}(-\text{sat}(\tilde{d}_\varepsilon(s)+d(s))+d(s))ds\end{aligned}$$

and $\|e^{\mathbb{A}t}\|\le M_1e^{-\omega t}$, we can obtain

$$\begin{aligned}\left\|\begin{pmatrix}X(t)\\ v(\cdot,t)\end{pmatrix}\right\|&\le M_1e^{-\omega t}\left\|\begin{pmatrix}X(0)\\ v(\cdot,0)\end{pmatrix}\right\|\\ &\quad+M_1e^{-\omega(t-T_0)}C_{T_0}T_0(2M+1)^2+LC_1\varepsilon,\end{aligned}$$

which leads to $\lim_{t\to\infty}\sup E_\varepsilon(t)\le C\varepsilon$, where $C>0$ is a constant independent of ϵ. ■

Remark 3.1: The time-varying, high-gain estimator can reduce the peaking value in the initial state, but it is not robust and amplifies the noise, whereas the constant high gain can filter the high frequency noise yet brings the peaking value problem. From practice point of view, a recommended control strategy is to use the time-varying gain first to reduce the peaking value in the initial stage to a reasonable level and then apply the constant high gain. This will be explained numerically in Section 5.

4. Sliding mode control

To compare with ADRC approach, we use the SMC to reject the disturbance $d(t)$ that satisfies $|d(t)|\le M$ for all $t\ge 0$ for some $M>0$.

We continue to consider system (2.6). First, set the sliding mode surface as

$$S^v=\left\{(X,h)\in\mathbb{H}:\int_0^1 xh(x)dx=0\right\},$$

which is a closed subspace of the state space $\mathbb{H}$. The corresponding sliding mode function for system (2.6) is

$$S_v(t)=\int_0^1 xv(x,t)dx. \tag{4.1}$$

On the sliding mode surface $S_v(t)\equiv 0$, system (2.6) becomes

$$\begin{cases}\dot{X}=(A+BK)X(t)+Bv(0,t),\ t\ge 0,\\ v_t(x,t)=v_x(x,t),\ x\in(0,1),\ t>0,\\ \int_0^1 xv(x,t)dx=0,\ t\ge 0,\\ X(0)=X_0,\\ v(x,0)=v_0(x),\ x\in[0,1].\end{cases} \tag{4.2}$$

Define the operator $\mathbf{A}$: $D(\mathbf{A})(\subset S^v)\to S^v$ as follows:

$$\begin{cases}\mathbf{A}[\mathbf{X},\mathbf{h}]^\top=[(\mathbf{A}+\mathbf{BK})\mathbf{X}+\mathbf{Bh}(\mathbf{0}),\mathbf{h}']^\top,\\ \quad\forall[X,h]^\top\in D(\mathbf{A}),\\ D(\mathbf{A})=\Big\{[X,h]^\top\in S^v\cap(\mathbb{R}^n\times H^1(0,1)):\\ \quad\int_0^1 xh'(x)dx=0\Big\}.\end{cases} \tag{4.3}$$

It is easy to show that for any $[X,h]^\top\in D(\mathbf{A})$, $\mathbf{A}[X,h]^\top\in S^v$ if and only if $h(1)=\int_0^1 h(x)dx$. Thus, we can rewrite $\mathbf{A}$ as

$$\begin{cases}\mathbf{A}[\mathbf{X},\mathbf{h}]^\top=[(\mathbf{A}+\mathbf{BK})\mathbf{X}+\mathbf{Bh}(\mathbf{0}),\mathbf{h}']^\top,\\ \quad\forall[X,h]^\top\in D(\mathbf{A}),\\ D(\mathbf{A})=\Big\{[X,h]^\top\in S^v\cap(\mathbb{R}^n\times H^1(0,1)):h(1)\\ \quad=\int_0^1 h(x)dx\Big\}.\end{cases} \tag{4.4}$$

The system (4.2) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}[X(t),v(\cdot,t)]^\top=\mathbf{A}[X(t),v(\cdot,t)]^\top,\\ &\quad t\ge 0,\ \text{with}\ [X(t),v(\cdot,t)]^\top=[X_0,v_0]^\top.\end{aligned}$$

Proposition 4.1: *Let $\mathbf{A}$ be defined by (4.4). Then $\mathbf{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$, consists of isolated eigenvalues of finitely algebraic multiplicity only. Moreover, $\mathbf{A}$ generates a C_0-semigroup on S^v. Further, $\sigma(\mathbf{A})=\sigma(A+BK)$ and $\sigma_{ess}(\mathbf{A})\subseteq\sigma(A+BK)$, where $\sigma_{ess}(\mathbf{A})$*

is the essential spectrum of $\mathbf{A}$*. Therefore, the system* (4.2) *associates with a* C_0*-semigroup of exponential stability on* S^v*.*

Proof: For any given $[X_1, h_1]^\top \in S^v$, solve

$$\mathbf{A}[X, h]^\top = [(A+BK)X + Bh(0), h']^\top = [X_1, h_1]^\top \tag{4.5}$$

to obtain

$$X = (A+BK)^{-1}\left(X_1 - 2B\int_0^1\int_0^x h_1(s)dsdx\right),$$
$$h(x) = -2\int_0^1\int_0^x h_1(s)dsdx + \int_0^x h_1(s)ds. \tag{4.6}$$

Hence, $\mathbf{A}^{-1}$ exists and is compact on S^v by the Sobolev embedding theorem (Adams & Fournier, 2003, p. 97). Therefore, $\sigma(\mathbf{A})$ consists of isolated eigenvalues of finitely algebraic multiplicity only (Müller, 2007, Theorem 13, p. 154).

Let inner product in S^v be induced by (2.15). Under this inner product, for any $Z = [X, h]^\top \in D(\mathbf{A})$, we have

$$\begin{aligned}
&\mathrm{Re}\langle \mathbf{A}Z, Z\rangle_1 \\
&= \frac{1}{2}\mathrm{Re}\left(X^\top(A+BK)^\top P\overline{X} + X^\top P(A+BK)\overline{X}\right) \\
&\quad + \mathrm{Re}(h(0)B^\top P\overline{X}) + \frac{\iota}{2}e^1|h(1)|^2 - \frac{\iota}{2}e^1|h(0)|^2 \\
&\quad -\frac{\iota}{2}\int_0^1 e^x|h(x)|^2dx \\
&\le -\frac{1}{2}|X|^2 - \frac{\iota}{2}\int_0^1 e^x|h(x)|^2dx + \frac{\iota}{2}e\int_0^1 |h(x)|^2dx \\
&\le \frac{e-1}{2}\|[X, h]^\top\|_1^2.
\end{aligned}$$

Hence, $\mathbf{A} - (e-1)/2I$ is dissipative. It follows from the Lumer-Phillips theorem (Pazy, 1983, Theorem 1.4.3) that $\mathbf{A} - (e-1)/2I$ generates a C_0-semigroup of contractions on S^v, and so $\mathbf{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$.

Next, we claim $\sigma(\mathbf{A}) = \sigma(A+BK)$. Obviously, $\sigma(A+BK) \subset \sigma(\mathbf{A})$ since if $\lambda_0 \in \sigma(A+BK)$ and X_0 is an eigenvector corresponding to λ_0, then $\lambda_0 \in \sigma(\mathbf{A})$ and a corresponding eigenfunction is given by $[X_0, 0]^\top$. On the other hand, for any $\lambda \notin \sigma(A+BK)$, $(\lambda I_{n\times n} - (A+BK))^{-1}$ exists, where $I_{n\times n}$ is the $n\times n$ identity matrix. For any given $[X_1, h_1]^\top \in S^v$, solve

$$\begin{aligned}
(\lambda I - \mathbf{A})[X, h]^\top &= [\lambda X - (A+BK)X - Bh(0), \lambda h - h']^\top \\
&= [X_1, h_1]^\top
\end{aligned} \tag{4.7}$$

to obtain

$$\begin{aligned}
X &= (\lambda I_{n\times n} - (A+BK))^{-1} \\
&\quad \times \left(X_1 - B\int_0^1 e^{\lambda(1-s)}h_1(s)ds \Big/ \int_0^1 e^{\lambda s}ds\right), \\
h(x) &= e^{\lambda x}\int_0^1 e^{\lambda(1-s)}h_1(s)ds \Big/ \int_0^1 e^{\lambda s}ds \\
&\quad - \int_0^x e^{\lambda(x-s)}h_1(s)ds,
\end{aligned} \tag{4.8}$$

which gives $\lambda \in \rho(\mathbf{A})$ if $\lambda \notin \sigma(A+BK)$. Therefore, $\sigma(\mathbf{A}) = \sigma(A+BK)$.

Now, we show that $\sigma_{\mathrm{ess}}(\mathbf{A}) \subseteq \sigma(A+BK)$. For this purpose, by the definition of essential spectrum (Engel & Nagel, 2000, P. 248), it suffices to prove that for any $\lambda \notin \sigma(A+BK)$, $\lambda I - \mathbf{A}$ is a Fredholm operator on S^v, which is equivalent to showing that (i) $\dim\ker(\lambda I - \mathbf{A}) < +\infty$, and (ii) $\dim(S^v/\mathrm{im}(\lambda I - \mathbf{A})) < +\infty$. Actually, assuming that $(\lambda I - \mathbf{A})[X, h]^\top = 0$, we have $(A+BK)X + Bh(0) = \lambda X, h' = \lambda h$. Thus, $h(x) = h(0)e^{\lambda x}$. By $\int_0^1 xh(x)dx = 0$, we can get $h(0) = 0$ and thus $h \equiv 0$. $X = 0$ follows from $\lambda \notin \sigma(A+BK)$. Therefore, $\ker(\lambda I - \mathbf{A}) = \{0\}$ and $\dim\ker(\lambda I - \mathbf{A}) < +\infty$. On the other hand, it follows from (4.7) and (4.8) that $\mathrm{im}(\lambda I - \mathbf{A})S^v = S^v$, which leads to $S^v/\mathrm{im}(\lambda I - \mathbf{A}) \simeq \{0\}$ and $\dim(S^v/\mathrm{im}(\lambda I - \mathbf{A})) < +\infty$. Thus, $\sigma_{\mathrm{ess}}(\mathbf{A}) \subseteq \sigma(A+BK)$. Since $A+BK$ is Hurwitz, it follows from Engel and Nagel (2000, Corollary 2.11) that $\|e^{\mathbf{A}t}\| \le Me^{\omega t}$, where $\omega = \max\{\sup\{\lambda : \lambda \in \sigma(\mathbf{A}), \sup\{\lambda : \lambda \in \sigma_{\mathrm{ess}}(\mathbf{A})\}\} < 0$, i.e.$e^{\mathbf{A}t}$ is exponentially stable. ■

Transforming S^v and S_v back to the original system (1.5) gives sliding mode surface

$$\begin{aligned}
S^w = \Bigg\{(X, h) \in \mathbb{H} : \int_0^1 x\Big[h(x) - \int_0^x k(x,y)h(y)dy \\
- Ke^{Ax}X\Big]dx = 0\Bigg\}
\end{aligned}$$

and the sliding function:

$$\begin{aligned}
S_w(t) := \int_0^1 xv(x,t)dt = \int_0^1 x\Bigg(w(x,t) \\
- \int_0^x k(x,y)w(y,t)dy - Ke^{Ax}X(t)\Bigg)dx.
\end{aligned} \tag{4.9}$$

On the sliding mode surface S^w, the original system (1.5) becomes

$$\begin{cases} \dot{X} = AX(t) + Bw(0,t), \ \ t \geq 0, \\ w_t(x,t) = w_x(x,t) + g(x)w(0,t) \\ \quad + \int_0^x f(x,y)w(y,t)dy, \ \ x \in (0,1), \ \ t \geq 0, \\ \int_0^1 x\left(w(x,t) - \int_0^x k(x,y)w(y,t)dy - Ke^{Ax}X(t)\right) \\ \quad dx = 0, \ \ t \geq 0, \\ X(0) = X_0, \\ w(x,0) = w_0(x), x \in (0,1). \end{cases} \tag{4.10}$$

It follows from the exponential stability of system (4.2) and the equivalence between (4.2) and (4.10) that system (4.10) is exponentially stable.

4.1 State feedback

To motivate the control design, differentiating (4.9) formally with respect to t, we obtain

$$\dot{S}_w(t) = U(t) + d(t) - \int_0^1 k(1,y)w(y,t)dy - Ke^{A}X(t) - \int_0^1 \left(w(x,t) - \int_0^x k(x,y)w(y,t)dy - Ke^{Ax}X(t)\right)dx.$$

If we choose an SMC as follows:

$$\begin{aligned} U(t) &= \int_0^1 \left(w(x,t) - \int_0^x k(x,y)w(y,t)dy - Ke^{Ax}X(t)\right)dx \\ &+ \int_0^1 k(1,y)w(y,t)dy + Ke^{A}X(t) - (M+\eta)\frac{S_w(t)}{|S_w(t)|}, \end{aligned} \tag{4.11}$$

where $\eta > 0$ is an any fixed constant, then we have

$$S_w(t)\dot{S}_w(t) \leq -\eta|S_w(t)|, \tag{4.12}$$

which is just the finite reaching condition. It is seen from (4.11) that SMC is always in worst-case concern where M is the up bound of the derivative of the disturbance.

Under the feedback control (4.11), the resulting closed-loop system is

$$\begin{cases} \dot{X} = AX(t) + Bw(0,t), \\ w_t(x,t) = w_x(x,t) + g(x)w(0,t) \\ \quad + \int_0^x f(x,y)w(y,t)dy, \\ w(1,t) = \int_0^1 \left(w(x,t) - \int_0^x k(x,y)w(y,t)dy \right. \\ \quad \left. - Ke^{Ax}X(t)\right)dx \\ \quad + \int_0^1 k(1,y)w(y,t)dy + Ke^{A}X(t) \\ \quad - (M+\eta)\frac{S_w(t)}{|S_w(t)|} + d(t), \\ X(0) = X_0, \\ w(x,0) = w_0(x). \end{cases} \tag{4.13}$$

Under the invertible transformation (2.3), the control (4.11) becomes

$$\begin{aligned} U(t) = \int_0^1 k(1,y)w(y,t)dy + Ke^{A}X(t) + \int_0^1 v(x,t)dx \\ - (M+\eta)\text{sign}(S_v(t)) \ \text{ for } S_v(t) \neq 0, \end{aligned} \tag{4.14}$$

and the closed loop of system (4.13) becomes

$$\begin{cases} \dot{X} = (A+BK)X(t) + Bv(0,t), \ \ t \geq 0, \\ v_t(x,t) = v_x(x,t), \ \ x \in (0,1), t \geq 0, \\ v(1,t) = \int_0^1 v(y,t)dy - (M+\eta)\frac{S_v(t)}{|S_v(t)|} + d(t) \\ \quad := \int_0^1 v(y,t)dy + \tilde{d}(t) \text{ for } S_v(t) \neq 0, \ \ t \geq 0, \\ X(0) = X_0, \\ v(x,0) = v_0(x), \ \ x \in [0,1]. \end{cases} \tag{4.15}$$

4.2 Properties of the closed loop

Lemma 4.1: *Suppose that the disturbance $d(t)$ is bounded measurable in time with the up bound M, for any initial value $[X_0, v_0]^\top \in \mathbb{H}$, there exists $T_{max} \geq 0$, depending on initial data, such that system (4.15) admits a unique solution $[X(t), v(\cdot,t)]^\top \in C(0, T_{\max}; \mathbb{H})$ and (4.15) will keep on the sliding mode surface $S_v(t) = 0$ for all $t \geq T_{max}$. Moreover, $S_v(t)$ is continuous and monotone in $[0, T_{max}]$. On the sliding mode surface $S_v(t) = 0$, and system (4.15) becomes (4.2) which is exponentially stable.*

Proof: Obviously, the sliding mode function (4.1) makes sense due to $v(\cdot, t) \in L^2(0,1)$. Similar to (2.7), system

(4.15) can be written as

$$\frac{d}{dt}[X(t), v(\cdot,t)]^\top = \mathbb{A}[X(t), v(\cdot,t)]^\top + \mathbb{B}\Big(\int_0^1 v(y,t)dy\Big) + \mathbb{B}\tilde{d}(t), \tag{4.16}$$

where $\mathbb{A}$ is given by (2.8) and $\mathbb{B} = [0, \delta(x-1)]^\top$. Next, we claim that for any $T > 0$, if $\tilde{d}(t) \in L^2(0,T)$, then (4.16) admits a unique solution $[X(t), v(\cdot,t)]^\top \in C(0,T;\mathbb{H})$. By Lemma 2.2, $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$, we have

$$\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds \in C(0,T;\mathbb{H}). \tag{4.17}$$

and for all $t > 0$,

$$\left\|\int_0^t e^{\mathbb{A}(\tau-s)}\mathbb{B}\zeta(s)ds\right\|_{\mathbb{H}} \le C_t\|\zeta\|_{L^2(0,t)} \le C_t t\|\zeta\|_{L^\infty(0,t)} \tag{4.18}$$

for some constant C_t which is independent of ζ. By Weiss (1989, Proposition 2.3), we know that C_t is nondecreasing with respect to t. Let $\tau_1 \le 1$. Then $C_{\tau_1} \le C_1$. Choose $\tau_1 > 0$ so that $C_1\tau_1 < 1$. Define the map $\mathbb{F}$ from $C(0,\tau_1;\mathbb{H})$ to $C(0,\tau_1;\mathbb{H})$ by

$$\mathbb{F}\begin{pmatrix} X(t) \\ v(\cdot,t)\end{pmatrix} = e^{\mathbb{A}t}\begin{pmatrix} X_0(\cdot) \\ v(\cdot,0)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\left(\int_0^1 v(y,t)dy\right)ds + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds. \tag{4.19}$$

It follows from (4.18) and (4.19) that for any $(X_1, v_1)^\top, (X_2, v_2)^\top \in C(0,\tau_1;\mathbb{H})$,

$$\begin{aligned}
&\left\|\mathbb{F}\begin{pmatrix} X_1(t) \\ v_1(\cdot,t)\end{pmatrix} - \mathbb{F}\begin{pmatrix} X_2(t) \\ v_2(\cdot,t)\end{pmatrix}\right\|_{\mathbb{H}} \\
&= \left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\left(\int_0^1 (v_1(y,s) - v_2(y,s)dy\right)ds\right\|_{\mathbb{H}} \\
&\le C_t t\left\|\int_0^1 (v_1(y,s) - v_2(y,s)dy\Big)ds\right\|_{L^\infty(0,t)} \\
&\le C_1\tau_1\left\|\int_0^1 (v_1(y,s) - v_2(y,s)dy\Big)ds\right\|_{L^\infty(0,\tau_1)} \\
&\le C_1\tau_1\|v_1(\cdot,s) - v_2(\cdot,s)\|_{L^\infty(0,\tau_1;L^2(0,1))} \\
&\le C_1\tau_1\left\|\begin{pmatrix} X_1(t) \\ v_1(\cdot,t)\end{pmatrix} - \begin{pmatrix} X_2(t) \\ v_2(\cdot,t)\end{pmatrix}\right\|_{C(0,\tau;\mathbb{H})},
\end{aligned} \tag{4.20}$$

which, together with $C_1\tau < 1$, implies that $\mathbb{F}$ is a strict contraction on $C(0,\tau_1;\mathbb{H})$. By the contraction mapping theorem, (4.19) has a unique fixed point $(X, v)^\top \in C(0,\tau_1;\mathbb{H})$, which is a solution of (4.16). Similarly, we can prove that (4.16) has a unique solution on an interval $[\tau_1, \tau_2]$, $\tau_1 < \tau_2$. Repeating the above process, we can show that (4.16) has a unique solution the maximal existence interval $[0, T]$.

Suppose that for some $T > 0$, $S_v(t) \neq 0$ for all $t \in [0, T]$. Take the inner product with $[0, x]^\top \in D(\mathbb{A}^*)$ on both sides of (4.16) to obtain

$$\begin{aligned}
\dot{S}_v(t) &= \frac{d}{dt}\int_0^1 xv(x,t)xdx \\
&= \frac{d}{dt}\langle [X(t), v(\cdot,t)]^\top, [0,x]^\top\rangle \\
&= \langle [X(t), v(\cdot,t)]^\top, \mathbb{A}^*[0,x]^\top\rangle \\
&\quad + \left(\int_0^1 v(x,t)dx + \tilde{d}(t)\right)\mathbb{B}^*x \\
&= -\int_0^1 v(x,t)dx + \left(\int_0^1 v(x,t)dx + \tilde{d}(t)\right) = \tilde{d}(t) \\
&= -(M+\eta)\frac{S_v(t)}{|S_v(t)|} + d(t),
\end{aligned} \tag{4.21}$$

that is,

$$\dot{S}_v(t) = -(M+\eta)\frac{S_v(t)}{|S_v(t)|} + d(t). \tag{4.22}$$

provided that $S_v(t) \neq 0$ and $S_v(t)$ is continuous in $t \in [0, T]$. So if (4.22) admits a unique continuous, nonzero solution, then (4.15) admits a unique solution $[X(t), v(\cdot,t)]^\top \in C(0,T;\mathbb{H})$.

Suppose that $T_0 \ge 0$ and $S_v(T_0) = S_0 \neq 0$. Then it follows from (4.22) that

$$S_v(t) = S_0 - (M+\eta)\int_{T_0}^t \frac{S_v(s)}{|S_v(s)|}ds + \int_{T_0}^t d(s)ds, \ \forall\, t \ge T_0. \tag{4.23}$$

Define a closed subspace of $C[T_0, T_0 + \frac{|S_0|}{5(2M+\eta)}]$ by

$$\Omega = \left\{S \in C\left[T_0, T_0 + \frac{|S_0|}{5(2M+\eta)}\right] : S(T_0) = S_0, |S(t)| \ge \frac{4}{5}|S_0|, \forall\, t \in \left[T_0, T_0 + \frac{|S_0|}{5(2M+\eta)}\right]\right\},$$

and the mapping F on Ω by

$$(FS)(t) = S_0 - (M+\eta)\int_{T_0}^t \frac{S_v(s)}{|S_v(s)|}ds + \int_{T_0}^t d(s)ds.$$

Then for any $S \in \Omega$,

$$|(FS)(t)| \geq |S_0| - (2M+\eta)(t-T_0) \geq \frac{4}{5}|S_0|,$$

which means that $F\Omega \subset \Omega$. Moreover,

$$\begin{aligned}
&|(FS_1)(t) - (FS_2)(t)| \\
&\leq (M+\eta)\left|\int_{T_0}^{t}\left[\frac{S_1(s)}{|S_1(s)|}ds - \frac{S_2(s)}{|S_2(s)|}\right]ds\right| \\
&\leq (M+\eta)\int_{T_0}^{t}\max\left\{\frac{1}{|S_1(s)|}, \frac{1}{|S_1(s)|}\right\} \\
&\quad \times |S_1(s) - S_2(s)|ds \\
&\leq \frac{M+\eta}{4(2M+\eta)}\|S_1 - S_2\|_\Omega,
\end{aligned} \tag{4.24}$$

where $\|S\|_\Omega = \max\{|S(t)| : t \in [T_0, T_0 + \frac{|S_0|}{5(2M+\eta)}]\}$. Since $(M+\eta)/(4(2M+\eta)) < 1$, (4.24) shows that the mapping F is contraction on Ω. By the Banach fixed point theorem, there exists a unique, nonzero solution $S_v(t)$ to (4.22) in $C[T_0, T_0 + \frac{|S_0|}{5(2M+\eta)}]$.

The above arguments show that when $S_v(0) \neq 0$, there exists a unique continuous solution $S_v(t)$ to (4.22) in the maximal interval $[0, T_{\max})$, where it must have $S_v(T_{\max}) = 0$. Notice that

$$\begin{aligned}
S_v(t)\dot{S}_v(t) &= S_v(t)\left[-(M+\eta)\frac{S_v(t)}{|S_v(t)|} + d(t)\right] \\
&\leq -\eta|S_v(t)|,
\end{aligned} \tag{4.25}$$

which is the finite reaching condition. This means that $|S_v(t)|$ must be decreasing in $[0, T_{\max})$ and $|S_v(t)| > 0$ for all $t \in [0, T_{\max})$. Since $S_v(t)$ is continuous, the reaching condition (4.25) implies that $S_v(t) = 0$ for all $t \geq T_{\max}$. This completes the proof of the lemma. ■

Returning to system (4.13) by inverse transformation (2.3), we obtain Theorem 4.1 since the closed-loop system (4.13) is equivalent to the closed-loop system (4.15).

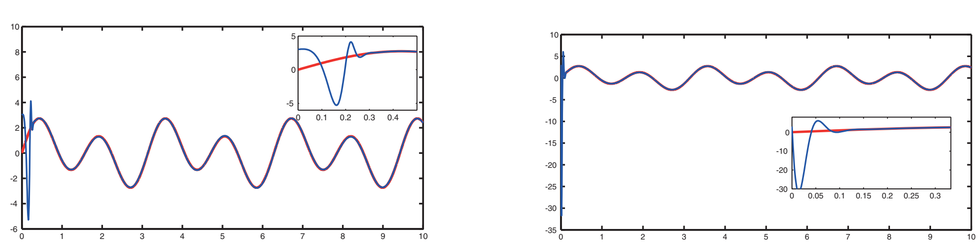
Theorem 4.1: *Suppose that the disturbance $d(t)$ is bounded measurable in time with the up bound M. Then, for any initial value $[X_0, w_0]^\top \in \mathbb{H}$, there exists $T_{max} \geq 0$, depending on initial data, such that system (4.13) admits a unique solution $[X(t), w(\cdot,t)]^\top \in C(0, T_{\max}; \mathbb{H})$ and (4.13) will keep on the sliding mode surface $S_w(t) = 0$ for all $t \geq T_{max}$. Moreover, $S_w(t)$ is continuous and monotone in $[0, T_{max}]$. On the sliding mode surface $S_w(t) = 0$, the system (4.13) becomes*

$$\begin{cases}
\dot{X} = AX(t) + Bw(0,t), \ t \geq 0, \\
w_t(x,t) = w_x(x,t) + g(x)w(0,t) \\
\quad + \displaystyle\int_0^x f(x,y)w(y,t)dy, \ x \in (0,1), \ t \geq 0, \\
\displaystyle\int_0^1 x\left(w(x,t) - \int_0^x k(x,y)w(y,s)dy - Ke^{Ax}X(t)\right) \\
\quad dx = 0, \ t \geq 0, \\
X(0) = X_0, \\
w(x,0) = w_0(x), x \in [0,1].
\end{cases} \tag{4.26}$$

which is exponentially stable.

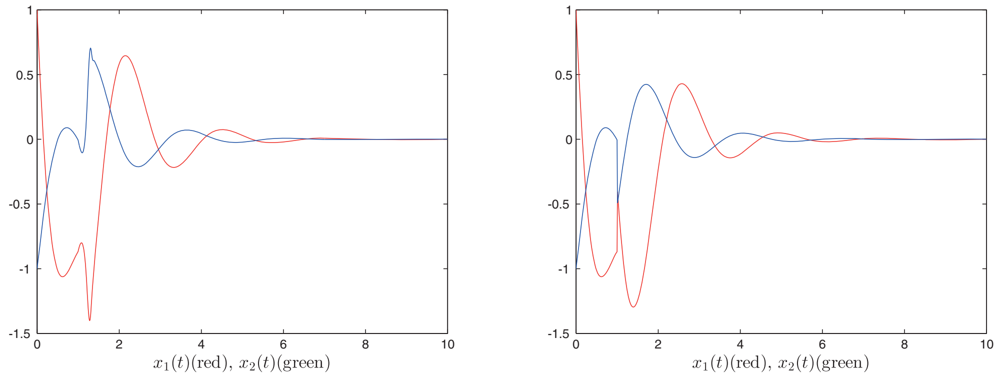
Remark 4.1: We compare the SMC approach with the ADRC for system (2.6). In the sliding mode design, the aim is to find a control such that the control can drive the state of system into the sliding surface $\int_0^1 xv(x,t)dx = 0$ at finite time, while the ADRC aims at estimating and cancelling the disturbance and making $v(1,t)$ as small as possible. The SMC is always in the worse-case concern and so it is robust to disturbance, whereas ADRC uses estimation/cancelation strategy and makes the control energy much small.

Remark 4.2: Here, SMC like in Guo and Jin (2013a), Guo and Liu (2014), Guo et al. (2014), and Wang et al. (2015) will cause chattering phenomenon, which can be seen from the next simulation section. It is well known that the chattering effect is significantly reduced by higher-order SMC (see Levant, 2005). Thus, to suppress the chattering, possibly, we need to design a higher-order sliding mode for PDE systems, like higher-order sliding mode for ODE system (Levant, 2005).



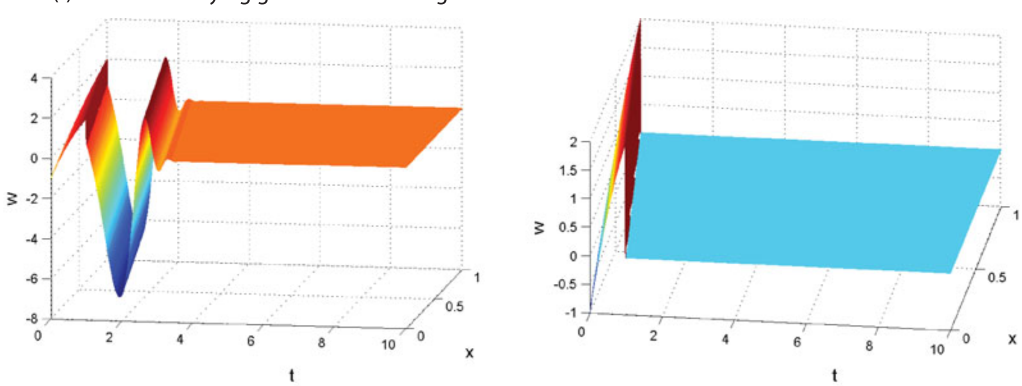
(a) d (red), $\hat{d}$ (blue) with the time varying gain (b) d (red), $\hat{d}$ (blue) with the constant gain

Figure 2. The disturbance $d(t)$ and its estimation with time-varying gain and constant gain.



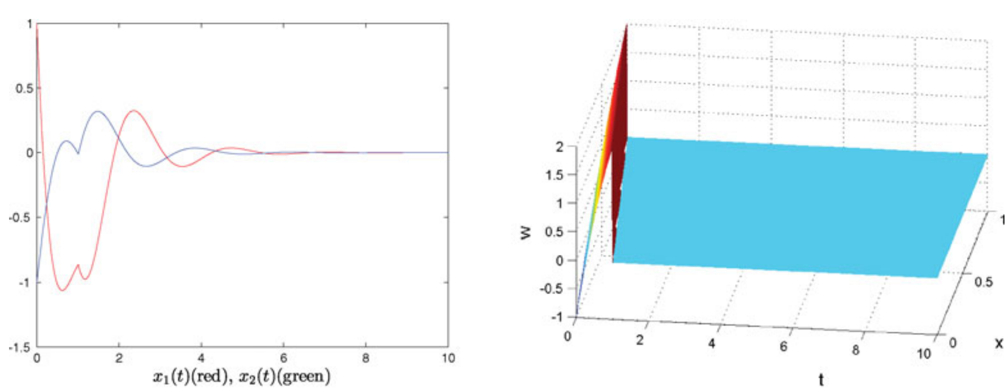
(a) The state $X(t)$ with the time varying gain (b) The state $X(t)$ with the constant gain

Figure 3. The state $X(t)$ with time-varying gain and constant gain.



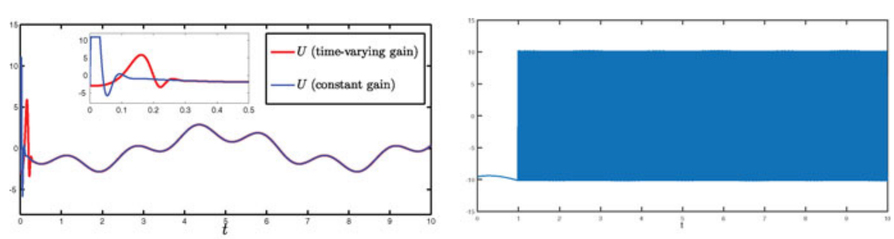
(a) The state $v(x,t)$ with the time varying gain (b) The state $v(x,t)$ with the constant gain

Figure 4. The state $v(x, t)$ with time-varying gain and constant gain.



(a) The state $X(t)$ (b) The state $v(x,t)$

Figure 5. The state $(X(t), v(x, t))$ by SMC.



(a) The control $U(t)$ by ADRC (b) The control $U(t)$ by SMC

Figure 6. The control $U(t)$ by ADRC and SMC.

5. Numerical simulation

In this section, a finite difference method is applied to compute numerically the displacement to illustrate the effectiveness of the proposed control. We notice that systems (3.11) and (4.15) are equivalent to (3.13) and (4.13), respectively. The numerical simulations for system (3.13) by ADRC and for system (4.15) by SMC are presented. The steps of space and time are taken as 0.05 and 0.005, respectively. In addition, we take

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$d(t) = 2\sin(4t) + \sin(2t),$$

and $K = \begin{pmatrix} -0.6 & 6.4 \end{pmatrix}$ so that the eigenvalues of $A + BK$ are $-1, -2$. The initial conditions are chosen as

$$X(0) = (1, -1)^\top,\ v(x, 0) = 4x - x^2 - 1,\ 0 \le x \le 1.$$

By ADRC, the function $\kappa(t)$ defined in (3.3) is taken as $\kappa(t) = 1/(1+t)$, and the time-varying gain is taken as

$$r(t) = \begin{cases} 1 + 19t, & t \in [0, 1], \\ 20, & t > 1. \end{cases}$$

The constant gain is $\epsilon = 0.05$ and the saturate function is taken as $\mathrm{sat}(x) = \min\{10, \max\{x, -10\}\}$. In ISMC, we take the design parameters as $M = 10, \eta = 0.2$.

Figure 2 plots the tracking errors for the disturbance by ADRC where Figure 2(a) is with the time-varying gain and Figure 2(b) is with the constant gain. It is clearly seen that the peaking value from Figure 2(b) is dramatically reduced by the time-varying gain in Figure 2(a). This is the merit of time-varying gain compared with the constant gain in Guo and Jin (2013a), Guo and Jin (2013b), and Guo and Zhao (2011).

Figures 3 and 4 display the states $X(t)$ and $v(x, t)$ of system (3.13) by ADRC with the time-varying and the constant gain, respectively. The convergence for both $X(t)$ and $v(x, t)$ is clearly seen.

Figure 5 displays the states $X(t)$ and $v(x, t)$ of system (4.15) by SMC. The convergence of $X(t)$ and $v(x, t)$ is clearly seen from Figure 5(a) and (b). We can also see that $v(x, t)$ arrives zero state at finite time, namely $v(x, t)$ is finite time stable.

Figure 6 displays the control $U(t)$ over time where Figure 6(a) is the control by ADRC and Figure 6(b) by SMC. It is obvious that the control $U(t)$ with time-varying gain tends to the same value of control $U(t)$ with constant gain. The control $U(t)$ with SMC has an infinity many switches.

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