Physics Letters B 768 (2017) 187–191

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Inelastic magnon scattering

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ARTICLE INFO

Article history:
Received 26 January 2017
Accepted 22 February 2017
Available online 28 February 2017
Editor: M. Cvetič

ABSTRACT

We study the worldsheet S-matrix of a string attached to a D-brane in $AdS_5 \times S^5$. The D-brane is either a giant graviton or a dual giant graviton. In the gauge theory, the operators we consider belong to the $su(2|3)$ sector of the theory. Magnon excitations of open strings can exhibit both elastic (when magnons in the bulk of the string scatter) and inelastic (when magnons at the endpoint of an open string participate) scattering. Both of these S-matrices are determined (up to an overall phase) by the $su(2|2)^2$ global symmetry of the theory. In this note we study the S-matrix for inelastic scattering. We show that it exhibits poles corresponding to boundstates of bulk and boundary magnons. A crossing equation is derived for the overall phase. It reproduces the crossing equation for maximal giant gravitons, in the appropriate limit. Finally, scattering in the $su(2)$ sector is computed to two loops. This two loop result, which determines the overall phase to two loops, will be useful when a unique solution to the crossing equation is to be selected.

1. Introduction

’t Hoofts original proposal that the large N expansions of Yang–Mills theories are equivalent to a string theory [1] is realized beautifully in the AdS/CFT correspondence [2]. Many concrete details of the duality can be confirmed with precision checks, thanks to integrability of planar $\mathcal{N}=4$ super Yang–Mills theory [3,4]: the planar diagrams give rise to a two dimensional effective theory which can be matched, in exquisite detail, to the worldsheet theory of a string. This detailed matching is possible because integrability allows the exact $\lambda = g_{YM}^2 N$ dependence of certain quantities to be computed.

There are many interesting string theory questions whose answers require the study of certain large N but non-planar limits of Yang–Mills theory. One such example is the study of the open string excitations of giant graviton branes. The physics of this problem requires summing many non-planar diagrams in the Yang–Mills theory, and so, corresponds to non-perturbative string effects [5]. Further, since the system is not in general integrable [6], a detailed comparison akin to what was achieved in the planar limit seems impossible. However, if one restricts to the $su(2|3)$ sector of the theory it turns out that the exact S matrix describing the scattering of worldsheet excitations can still be determined up to an overall phase, by making use of the global $su(2|2)^2$ symmetry enjoyed by this sector [7,8]. The scattering of magnon excitations of open strings is inelastic [6], which is a strong hint that the system is not integrable.

The fact that some quantities can be computed exactly, even without integrability, is extremely interesting and deserves to be explored in detail. The first goal of this study is to explore the structure of the S-matrix for inelastic magnon scattering and verify that it has the structure we expect. Specifically, the analyticity and unitarity of the S-matrix imply a correspondence between singularities of the S-matrix and on-shell intermediate states. This is the subject of section 2. We find a pole corresponding to binding a bulk and a boundary magnon. The structure of boundstates that we uncover smoothly interpolates between the bound state structure of bulk magnons [9,10] (for small giant gravitons when $r \approx 1$) and the bound state structure obtained for maximal giants [11] (when $r \approx 0$). The boundstate is a BPS state in the double box representation of $su(2|2)^2$. The second goal of this work is to study the overall phase of the S-matrix. This phase is constrained by a crossing symmetry equation [12,13]. Using insights following from similar studies of the same question in the planar limit [8,11], we write down an equation obeyed by this phase, by considering the scattering of a magnon with a singlet state. Although we have not managed to solve this equation, we have checked that it reduces the crossing equation [11] obtained for maximal giant gravitons in an appropriate limit. Finally, we study scattering in the $su(2)$

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http://dx.doi.org/10.1016/j.physletb.2017.02.056

sector, perturbatively to two loops, in the super Yang–Mills theory. These results can be used to single out a unique solution to the crossing equation.

2. Bound state spectrum

The scattering problem is most conveniently described using complex spectral parameters $x^\pm$. In terms of these parameters, the charge, energy and momentum of a magnon can be written as follows [6]

$$n = \frac{g}{i}\left(x^+ + \frac{1}{x^+} - rx^- - \frac{r}{x^-}\right) \tag{2.1}$$

$$E = \frac{g}{i}\left(x^+ - \frac{1}{x^+} - rx^- + \frac{r}{x^-}\right) \qquad e^{ip} = \frac{x^+}{x^-} \tag{2.2}$$

where $r = 1$ for a bulk magnon, $0 \le r < 1$ for a boundary magnon attached to a giant graviton and $r > 1$ for a boundary magnon attached to a dual giant graviton. Using the above relations we can determine the energy of a magnon in terms of its charge and momentum as

$$E = \sqrt{n^2 + 4g^2(1+r^2) - 8g^2 r\cos(p)} \tag{2.3}$$

The condition that $x^+ - rx^-$ and $\frac{1}{x^+} - \frac{r}{x^-}$ are pure imaginary will ensure real charges, energies, and momenta. To look for boundstates we will analytically continue the spectral parameters by relaxing this condition, allowing the energy and momenta to be complex. We will however maintain (2.1): this is the condition to have a short (atypical) representation of $su(2|2)^2$. It is only for these representations that the tensor product of two representations is irreducible and hence that the $su(2|2)^2$ symmetry is sufficient to fix the S-matrix up to an overall phase. The inverse relation is

$$x^\pm = \frac{ie^{\pm i\frac{p}{2}}(E+n)}{2g(e^{i\frac{p}{2}} - re^{-i\frac{p}{2}})} \tag{2.4}$$

The scattering of a bulk and a boundary magnon is inelastic, as we now explain. We use a subscript 1 to denote the bulk magnon before scattering and a subscript 2 to denote a boundary magnon before scattering. We used primed subscripts for the magnons after scattering. The momenta, energies and charges after scattering are determined by solving

$$E_1 + E_2 = E_1' + E_2' \qquad p_1 + p_2 = p_1' + p_2' \qquad n_1 + n_2 = n_1' + n_2' \tag{2.5}$$

These equations can be reduced to the solution of a cubic equation that has a single real root, but the details are not very illuminating. Close to $r = 0$ and $r = 1$ we can do better though: at $r = 1 - \epsilon$ we have

$$x_1'^\pm = x_2^\pm + \delta x_1'^\pm \qquad x_2'^\pm = x_1^\pm + \delta x_2'^\pm \tag{2.6}$$

while at $r = \epsilon$ we have

$$x_1'^\pm = -x_1^\mp + \delta x_1'^\pm \qquad x_2'^+ = x_2^+ + \delta x_2'^+$$
$$x_2'^- = \left(\frac{x_1^-}{x_1^+}\right)^2 x_2^- + \delta x_2'^- \tag{2.7}$$

Working to order ϵ we find a set of linear equations whose explicit solution shows that $\delta x_1'^\pm \sim O(\epsilon)$, $\delta x_2'^\pm \sim O(\epsilon)$ for both cases. The linear equations we use arise by assuming we scatter elementary magnons (so that $n_1 = n_2 = n_1' = n_2' = 1$), as well as energy and momentum conservation. The fact that $\delta x_1'^\pm$, $\delta x_2'^\pm$ are non-zero is a clear indication that the scattering is not elastic.

We now focus attention on the scattering of magnons that belong to the $su(2)$ sector of the theory. This is still perfectly general since the global symmetry of the theory then determines scattering in any other sector [7]. In this case, up to an undetermined overall phase, the S-matrix is given by[1]

$$R|\phi_1^1\phi_2^1\rangle = A_{12}^R|\phi_{1'}^1\phi_{2'}^1\rangle \tag{2.8}$$

where

$$\begin{aligned}A_{12}^R = R_{12}^0\big[\eta_1\eta_2 x_1'^+ x_1^+(x_1^- - x_2^+)\big((x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-)x_2^+ \\ + (x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)x_2'^+\big)\big]\big[\eta_1'\eta_2' x_2'^+ x_2^+(x_1^- - x_1^+) \\ \times (x_1^+ - x_1'^+)(x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+))\big]^{-1}\end{aligned} \tag{2.9}$$

This reduces to the correct bulk [7] and reflection [11] matrices when we set $r = 1$ and $r = 0$ respectively.[2] The statement that the S-matrix is unitary is the statement

$$A_{12}^R A_{1'2'}^R = 1 \tag{2.10}$$

which we have verified holds for any r, as it should.

We will now look for singularities in the S-matrix. The presence of simple poles indicates on shell intermediate states. Investigation of the singularities of the elastic magnon S-matrix has uncovered a wealth of BPS boundstates [9,10,15]. We will argue below that we find an equally rich spectrum of BPS boundstates that naturally interpolates between the boundstates of bulk magnons [9] and the boundstates of a bulk magnon and a boundary magnon associated to a maximal giant graviton [11]. Inspection of (2.9) suggests possible poles when $x_2'^+ = 0$ or when $x_2^+ = 0$. Since the charges n_k are positive integers and since we want to keep $Re(E_k) > 0$, its clear from (2.4) that these poles can't be realized. The factor in the denominator $(x_1^- - x_1^+)(x_1^+ - x_1'^+)$ can also give rise to a pole. Analyzing this factor near $r = 1$ we find a pole at

$$x_1^+ = x_2^+ + 2(1-r)x_2^+\frac{x_2^+x_2^- - 1}{(x_2^+ - \frac{1}{x_2^+})(x_2^+ - x_2^-)} + O\left((1-r)^2\right) \tag{2.11}$$

This is precisely canceled, by a zero coming from the factor $(x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-)x_2^+ + (x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)x_2'^+$ in the numerator. Near $r = 0$ the factor $(x_1^- - x_1^+)(x_1^+ - x_1'^+)$ in the denominator leads to a pole at

$$x_1^- = -x_1^+ + r\frac{x_1^+}{x_2^-}\frac{(x_1^+ - \frac{1}{x_1^+})(x_2^- - x_2^+)(x_2^- + x_2^+)}{x_2^-(x_2^+ - \frac{1}{x_2^+})} + O\left(r^2\right) \tag{2.12}$$

This is again canceled, by a zero coming from the factor $(x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-)x_2^+ + (x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)x_2'^+$ in the numerator. Thus, in the end we find that a single pole arises when $x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+) = 0$, which implies that

$$x_1^+ = x_2^-\frac{x_2^+ - rx_2^-}{rx_2^+ - x_2^-} \tag{2.13}$$

To interpret this pole recall that singularities of the S-matrix correspond to spacetime diagrams where each particle is on-shell [16].

[1] Here, following [6], we use the notation R to denote the S-matrix for the scattering of a bulk and a boundary magnon. We reserve S for the S-matrix of bulk magnon scattering, which is an elastic process.

[2] To make this comparison we found [14] very useful.

Particle worldlines meet at vertices which conserve charge, energy and momentum. We want to consider a cubic vertex corresponding to the creation of a boundstate from a boundary and a bulk magnon. Using b' to denote the boundstate of a boundary and a bulk magnon, the conservation of charge, energy and momentum implies that

$$E(x_1^\pm) + E(x_2^\pm) = E(x_{b'}^\pm)$$
$$p(x_1^\pm) + p(x_2^\pm) = p(x_{b'}^\pm)$$
$$n(x_1^\pm) + n(x_2^\pm) = n(x_{b'}^\pm) \tag{2.14}$$

Since this is three equations we can completely determine the spectral parameters of the boundstate using two of the above equations. The third equation then implies a relation between the magnon and boundary magnon spectral relations that is obeyed by (2.13). It is straightforward to apply the rules described in [15] and verify that this pole signals are a normalizable wave function, for a boundstate with charge $n = 2$ and energy given by (2.3) evaluated at this n. Further, when $r = 1$ (2.13) is the pole identified in [9] as a signal of a bound state of bulk magnons and when $r = 0$ (2.13) is the pole identified in [11] as the signal of a bound state of a bulk and a boundary magnon, for the case of open string attached to a maximal giant graviton.

We can now continue and consider the scattering of a bulk magnon with this boundate, producing a new boundstate with charge $n = 3$. Indeed, following the rules of [15], we find a family of boundstate with one boundary magnon bound to $n - 1$ bulk magnons. This boundstate has a charge of n and energy given by (2.3). By varying r smoothly from $r = 0$ to $r = 1$, this structure of boundstates nicely interpolates from the known structure of boundstates when bulk magnons bind to the boundary magnon of a maximal giant graviton [11] and the known structure for bulk magnons boundstates [9].

3. Crossing equation

To derive the crossing equation, we will follow the derivation given in [8]. The same method has been applied to determine the crossing equation for maximal giant gravitons in [11]. The idea is to consider the scattering from the singlet state

$$|I_{\bar{p},p}\rangle = f_p\left(|\phi^1_{\bar{p}}\phi^2_p\rangle - |\phi^2_{\bar{p}}\phi^1_p\rangle\right) + |\psi^1_{\bar{p}}\psi^2_p\rangle - |\psi^2_{\bar{p}}\psi^1_p\rangle \tag{3.1}$$

The phase factor f_p is determined by requiring that the singlet state is annihilated by all of the $su(2|2)$ elements, as explained in [8]. To obtain the crossing equation, we consider the scattering of this impurity from the right boundary magnon. The only difference we find as compared to [11], is that the scattering is inelastic. Since the scattering is inelastic, we find

$$R(\bar{p}, q')S(\bar{p}, p')R(p, q)|I_{\bar{p},p}, q\rangle = \phi_r|I_{p',\bar{p}'}, q\rangle \tag{3.2}$$

where r is a phase. If we now scatter the singlet from the left boundary magnon, explicit computations show that we pick up the same phase, so that we return to the original state apart from the phase ϕ_r^2. The crossing equation is now obtained by requiring that ϕ_r^2 is one, i.e. that $\phi_r = \pm 1$. To choose the correct sign, we compare to the $r = 0$ limit (studied in [11]) and conclude that we should impose $\phi_r = 1$. Since this crossing equation involves both the scattering of bulk with bulk magnons and the scattering of boundary with bulk magnons, it relates the overall phase factor of S (which has been determined) to the overall phase factor of R (what we want to determine). After a tedious computation we find the following result

$$\left[\left(f_p C^R(p,q)G^S(\bar{p},p') - 2L^R(p,q)K^S(\bar{p},p')\right)\left(H^R(\bar{p},q')K^R(\bar{p},q') - G^R(p,q')L^R(\bar{p},q')\right)\right]\left[2L^R(\bar{p},q')\right]^{-1} = 1 \tag{3.3}$$

where the matrix elements of R (denoted with superscript R) are derived in [6] and the matrix elements of S (denoted with superscript S) are derived in [8]. We will refer to the function on the LHS of (3.3) as the crossing function. Scattering with different boundary magnon states determines crossing functions that have a different expression in terms of the matrix elements of R and S, but lead to the same crossing equation. We have not written this crossing equation in terms of the spectral parameters of the initial and final magnons as these expressions are rather long. The phase we have discussed above arises from an $su(2|2)$ factor. The theory actually enjoys $su(2|2)^2$ symmetry, so the full reflection factor is the square of the phase factor we discussed above.

As a first check of these results, we note that when $r = 0$ they reproduce the crossing equation quoted in equation (41) of [17]. In addition to this, a numerical study of the crossing equation reveals an appealing symmetry. Recall that we denote the boundary magnon momentum by q and the bulk magnon momentum by p. The crossing equation obtained from scattering off the right boundary with momenta $p + q \to p' + q'$ is identical to crossing equation obtained the left scattering with the same momenta $q + p \to q' + p'$. If one considers right scattering with momenta $p' + q' \to p + q$ or left scattering with momenta $q' + p' \to q + p$ the crossing equations are again identical: for all four situations we obtain the same crossing equation. The appearance of this symmetry is important for the consistency of our derivation of the crossing equation since it ensures that scattering the singlet from both boundaries as described above, does indeed return us to our initial state. It is satisfying that this derivation of the crossing equation works even though we have inelastic scattering.

4. *su*(2) scattering to two loops

We will now consider the scattering of a bulk and a boundary magnon, at two loops, in the super Yang–Mills theory. This will allow us to determine the overall phase of R to two loops, which will be useful data when a unique solution to the crossing equation is to be singled out. The operators in the Yang–Mills theory dual to giant gravitons are given by Schur polynomials [18–20]. Giant gravitons with open string excitations are dual to the restricted Schur polynomials, constructed in [21,22]. The action of the dilatation operator on the restricted Schur polynomials has been constructed in [23,24,6]. In what follows below we use this action at two loops to define the Hamiltonian for a Schrödinger equation based description of the magnon scattering. The Bethe ansatz for the wave function is given by [25]

$$\psi(l_1, l_2) = e^{ip_1l_1 + ip_2l_2} + Ae^{ip'_1l_1 + ip'_2l_2} + g^2\phi(l_1)\delta_{|l_1-l_2|,1} \tag{4.1}$$

where if $|l_1 - l_2| > 2$ the wave function must obey the Schrödinger equation

$$\begin{aligned}
&E\Psi(l_1,l_2)\\
&= g^2(3+r^2)\Psi(l_1,l_2) - g^2r\left(\Psi(l_1-1,l_2)+\Psi(l_1+1,l_2)\right)\\
&\quad - g^2\left(\Psi(l_1,l_2-1)+\Psi(l_1,l_2+1)\right)\\
&\quad - g^4\left(\left(1+r^2\right)^2+4\right)\Psi(l_1,l_2)\\
&\quad + 2g^4\left(1+r^2\right)r\left(\Psi(l_1-1,l_2)+\Psi(l_1+1,l_2)\right)\\
&\quad + 4g^4\left(\Psi(l_1,l_2-1)+\Psi(l_1,l_2+1)\right)
\end{aligned}$$

$$
\begin{aligned}
&- g^4 r^2 \left(\Psi(l_1-2,l_2) + 2\Psi(l_1,l_2) + \Psi(l_1+2,l_2)\right) \\
&- g^4 \left(\Psi(l_1,l_2-2) + 2\Psi(l_1,l_2) + \Psi(l_1,l_2+2)\right), \qquad (4.2)
\end{aligned}
$$

l_1 is the position of the boundary magnon, with momentum p_1. l_2 is the position of the bulk magnon, with momentum p_2. If $l_2 = l_1 + 2$, the Schrödinger equation becomes

$$
\begin{aligned}
&E\Psi(l_1,l_1+2) \\
&= g^2(3+r^2)\Psi(l_1,l_1+2) - g^2 r \left(\Psi(l_1-1,l_1+2)\right. \\
&\quad \left. + \Psi(l_1+1,l_1+2)\right) - g^2\left(\Psi(l_1,l_1+1) + \Psi(l_1,l_1+3)\right) \\
&\quad - g^4 \left(\left(1+r^2\right)^2 + 4\right)\Psi(l_1,l_1+2) \\
&\quad + 2g^4\left(1+r^2\right) r \left(\Psi(l_1-1,l_1+2) + \Psi(l_1+1,l_1+2)\right) \\
&\quad + 4g^4\left(\Psi(l_1,l_1+1) + \Psi(l_1,l_1+3)\right) \\
&\quad - g^4\left(4\Psi(l_1,l_1+2) + \Psi(l_1,l_1+4)\right) \\
&\quad - g^4 r^2\left(\Psi(l_1-2,l_1+2) + 2\Psi(l_1,l_1+2)\right), \qquad (4.3)
\end{aligned}
$$

and if $l_2 = l_1 + 1$, the Schrödinger equation becomes

$$
\begin{aligned}
&E\Psi(l_1,l_1+1) \\
&= g^2(1+r^2)\Psi(l_1,l_1+1) - g^2 r\Psi(l_1-1,l_1+1) \\
&\quad - g^2\Psi(l_1,l_1+2) - g^4\left(r^4+1\right)\Psi(l_1,l_1+1) \\
&\quad + 2g^4\left(1+r^2\right) r\Psi(l_1-1,l_1+1) + 4g^4\Psi(l_1,l_1+2) \\
&\quad - g^4 r^2\left(\Psi(l_1-2,l_1+1) + \Psi(l_1,l_1+1)\right) \\
&\quad - g^4\left(\Psi(l_1,l_1+1) + \Psi(l_1,l_1+3)\right) \\
&\quad - g^4 r\left(\Psi(l_1-1,l_1) + \Psi(l_1+1,l_1+2)\right), \qquad (4.4)
\end{aligned}
$$

From (4.2) we learn that

$$
\begin{aligned}
E &= g^2(3+r^2) - g^2 r(e^{-ip_1} + e^{ip_1}) - g^2(e^{-ip_2} + e^{ip_2}) \\
&\quad - g^4(1+r^2)^2 - 4g^4 + 2g^4(1+r^2) r(e^{-ip_1} + e^{ip_1}) \\
&\quad + 4g^4(e^{-ip_2} + e^{ip_2}) - g^4(e^{-2ip_2} + 2 + e^{2ip_2}) \\
&\quad - g^4 r^2(e^{-2ip_1} + 2 + e^{2ip_1}) \qquad (4.5)
\end{aligned}
$$

From (4.3) we find

$$
\begin{aligned}
r\phi(l_1+1) + \phi(l_1) &= r^2\psi(l_1+2,l_1+2) + \psi(l_1,l_1) \\
&\quad - 2\psi(l_1,l_1+2) \qquad (4.6)
\end{aligned}
$$

Finally, from (4.4) we find

$$
\begin{aligned}
&g^4(2 - r(e^{-ip_1} + e^{ip_1}) - e^{ip_2} - e^{-ip_2})\phi(l_1) \\
&= -g^2(g^2 r\psi(l_1-1,l_1) - g^2\psi(l_1,l_1-1) - \psi(l_1,l_1) \\
&\quad + 4g^2\psi(l_1,l_1) + 2\psi(l_1,l_1+1) - 5g^2\psi(l_1,l_1+1) \\
&\quad - 3g^2 r^2\psi(l_1,l_1+1) - r\psi(l_1+1,l_1+1) \\
&\quad + 2g^2 r\psi(l_1+1,l_1+1) + 2g^2 r^3\psi(l_1+1,l_1+1) \\
&\quad + g^2 r\psi(l_1+1,l_1+2) - g^2 r^2\psi(l_1+2,l_1+1)) \qquad (4.7)
\end{aligned}
$$

Starting from (4.7) we are able to solve for $\phi(l_1)$. We find that $\phi(l_1)$ is independent of l_1 which is intuitively appealing. Inserting the solution for $\phi(l_1)$ into (4.6), we are able to solve for A. The result is

$$
\begin{aligned}
A &= -\frac{1 - 2e^{ip_2} + e^{i(p_1+p_2)} r}{1 - 2e^{ip_2'} + e^{i(p_1'+p_2')} r} \\
&\quad - \frac{g^2(e^{ip_1} - e^{ip_1'})}{(1 + re^{i(p1+p2)})(1 - 2e^{ip_2'} + re^{i(p_1'+p_2')})^2} \\
&\quad \times \left[2\left(re^{i(p_1+p_2)} - 2e^{ip_2} + 1\right)\right. \\
&\quad \times \left(re^{i(2p_1+p_2)} - 2e^{i(p_1+p_2)} + e^{i(p_1+2p_2)} + e^{ip_1} + e^{ip_2} r\right) \\
&\quad \times e^{i(-p_1'-p_1+p_2')} \\
&\quad + e^{i(-p_1'-p_1)}\left(-e^{ip_2}\left(e^{ip_1}\left(r^2-9\right) + 4e^{2ip_1} r + 2r\right)\right. \\
&\quad - e^{2ip_2}\left(4e^{3ip_1} r^2 + e^{2ip_1}\left(r^2-7\right) r + 6e^{ip_1} - 2r\right) \\
&\quad + e^{i(p_1+3p_2)}\left(2 + r\left(-4e^{3ip_1} r^2 + e^{2ip_1}\left(r^2+7\right) r\right.\right. \\
&\quad - \left.\left. 2e^{ip_1}\left(r^2+2\right) + 2r\right)\right) \\
&\quad + \left. re^{2i(p_1+2p_2)}\left(2 + e^{ip_1} r\left(-2 + e^{ip_1} r\right)\right) - 4e^{ip_1} + r\right) \\
&\quad + \left(1 + re^{i(p_1+p_2)}\right)\left(re^{i(p_1+p_2)} - 2e^{ip_2} + 1\right) \\
&\quad \times \left.\left(r^2 e^{i(p_1+p_2)} + e^{i(-p_1-p_2)}\right)\right] \qquad (4.8)
\end{aligned}
$$

Recall that total R-matrix has a contribution from each of the $su(2|2)$ factors, so that

$$
R_{\mathrm{R}}(x_1,x_2,x_1',x_2') = R_{su(2|2)}(x_1,x_2,x_1',x_2') \otimes R_{su(2|2)}(x_1,x_2,x_1',x_2') \qquad (4.9)
$$

Consequently, setting A in (4.8) to be equal to $(A_{12}^R)^2$ with A_{12}^R given in (2.9), we determine the overall phase to two loops.

Acknowledgements

This work is based upon research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the NRF and DST do not accept any liability with regard thereto.

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